Houston Journal of Mathematics

Volume 48, Number 3, 2022, Pages 631–653

GLOBAL EXISTENCE FOR QUASILINEAR WAVE EQUATIONS SATISFYING THE NULL CONDITION

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Communicated by Shiferaw Berhanu

Abstract. We explore the global existence of solutions to systems of quasilinear wave equations satisfying the null condition when the initial data are sufficiently small. We adapt an approach of Keel, Smith, and Sogge, which relies on integrated local energy estimates and a weighted Sobolev estimate that yields decay in $|x|$, by using the r^p-weighted local energy estimates of Dafermos and Rodnianski. One advantage of this approach is that all time-dependent vector fields can be avoided and the proof can be readily adapted to address wave equations exterior to star-shaped obstacles.

1. Introduction

This article focuses on re-examining the proof of small data global existence for systems of wave equations satisfying the classical null condition in $(1 + 3)$-dimensions. The proof relies only on the translational and rotational symmetries of the d'Alembertian. No explicit decay in time is required. Instead, in the spirit of the almost global existence proofs of [5] and [14], a weighted Sobolev estimate that provides decay in $|x|$ is paired with a local energy estimate. In this case, however, for the "good" derivatives that the null condition promises, we use the r^p-weighted local energy estimate of [2]. When considering quasilinear equations one in essence has geometry that depends on the solution while the solution in turn depends on the geometry. The highest order estimates need to be adapted to this geometry. Upon performing the typical manipulations for

2020 *Mathematics Subject Classification.* 35L71; 35L05.

Key words and phrases. Quasilinear wave equations, null condition, global existence, local energy estimate.

The second author was supported in part by Simons Foundation Collaboration Grant 711724 and NSF grant DMS-2054910.

the r^p-weighted estimates, the possibility that the "good" derivative from the multiplier and the "good" derivative from the null condition land on the same factor is encountered. Our method introduces a simple approach to this issue, which [7] calls "the problem of multiple good derivatives", by allowing for different choices of p in the r^p-weighted estimate.

Specifically, we will consider

$$\begin{aligned} \Box u^I &= A^{I,\alpha\beta}_{JK}\partial_\alpha u^J \partial_\beta u^K + B^{I,\gamma\alpha\beta}_{JK}\partial_\gamma u^J \partial_\alpha\partial_\beta u^K, \\ u(0,\,\cdot\,) &= f, \quad \partial_t u(0,\,\cdot\,) = g. \end{aligned} \tag{1.1}$$

Here $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3$, $I = 1, 2, \ldots, M$, and $u = (u^1, \ldots, u^M)$. Repeated Greek indices are implicitly summed from 0 to 3 where $\partial_0 = \partial_t$, lower case Latin indices are summed from 1 to 3, and repeated upper case indices are summed from 1 to M. The coefficients of the quasilinear terms are assumed to satisfy the symmetries

$$B^{I,\gamma\alpha\beta}_{JK} = B^{I,\gamma\beta\alpha}_{JK} = B^{K,\gamma\alpha\beta}_{JI}. \tag{1.2}$$

We have truncated (1.1) at the quadratic level. As is well-known, for problems with small initial data, higher order terms are typically better behaved.

Even for small, sufficiently regular and decaying initial data, solutions to (1.1) can only be ensured to exist almost globally, which means that the lifespan of the solution grows exponentially as the size of the initial data shrinks. In [1] and [8], the *null condition* was identified as a sufficient condition for guaranteeing global solutions to (1.1) for small initial data. We assume the same here, which requires that

$$A^{I,\alpha\beta}_{JK}\xi_\alpha\xi_\beta = 0, \text{ and } B^{I,\gamma\alpha\beta}_{JK}\xi_\alpha\xi_\beta\xi_\gamma = 0, \text{ whenever } \xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 = 0. \tag{1.3}$$

These conditions promise that at least one factor of each nonlinear term is a "good" derivative, which are directional derivatives in directions that are tangent to the light cone $t = |x|$ and are known to have more rapid decay. We will fix the notation and more explicitly describe these in the next subsection.

The main result of this paper establishes global existence for (1.1) subject to (1.3) for sufficiently small initial data.

Theorem 1.1. *Fix $0 < \tilde{p} < 2$, and suppose $f, g \in C^\infty(\mathbb{R}^3)$ satisfy*

$$\sum_{|\mu|\le N} \Big(\|\langle r\rangle^{\frac{\tilde{p}}{2}+|\mu|}\partial^\mu \nabla f\|_{L^2} + \|\langle r\rangle^{\frac{\tilde{p}}{2}+|\mu|}\partial^\mu g\|_{L^2} + \|\langle r\rangle^{\frac{\tilde{p}-2}{2}+|\mu|}\partial^\mu f\|_{L^2}\Big) \le \varepsilon \tag{1.4}$$

for $\varepsilon > 0$ sufficiently small and N sufficiently large. Then provided that (1.2) *and* (1.3) *hold,* (1.1) *admits a global solution $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$.*

Here we have used $\langle r\rangle = \sqrt{1+r^2}$. We note that the assumption (1.4) could be refined after we introduce some additional notation.

A common approach in proofs of long-time existence for nonlinear wave equations is to rely on the method of invariant vector fields and the Klainerman-Sobolev inequality [9]. Motivated by a desire to study similar wave equations in exterior domains, with multiple speeds, and with nontrivial background geometry, multiple approaches now exist that do not necessitate the use of the full set of invariant vector fields. A sample of such results includes [16], [6], [15, 13], [11, 12], [10].

The majority of the above results still rely on the scaling vector field $t\partial_t + r\partial_r$. Works such as [4], [7], [18] have pioneered methods for quasilinear equations that do not rely on any time dependent vector field. These methods rely on other means of obtaining t-decay. In the case of, e.g., [7] this is accomplished by considering a null foliation.

In the current paper, we explore a technique that is more akin to [5], [14]. Rather than relying on decay in t, this approach couples decay in $|x|$ with local energy estimates for the wave equation in order to obtain almost global existence without assuming special structures on the nonlinearity. In order to take advantage of the good derivatives that the null condition ensures, we will employ the r^p-weighted local energy estimate of [2]. We note that our method can immediately be adapted to prove the same result for Dirichlet-wave equations exterior to star-shaped obstacles.

In [3], this same approach was explored for semilinear wave equations. The current result is a bit more involved. In order to avoid a loss of regularity at the highest order, the estimates need to be adapted to allow for small, time-dependent perturbations of $\Box$. Upon doing so, "the problem of multiple good derivatives" as described in [7, Section 1.5.4] is encountered. In a typical term encountered within the r^p-weighted local energy estimate, there is a cubic interaction. Two factors arise from the quadratic nonlinearity and one from the multiplier. Amongst these factors, the multiplier contributes a good derivative and the null condition promises at least one additional good derivative. If these good derivatives fall on different factors, the method of [3] applies easily. When adapting the estimates to allow for the perturbations, the manipulations allow for these good derivatives to both fall on the same factor. We propose an alternative to the methods of [7] for subverting this problem. In particular, we consider separately a lower order energy and a high order energy, which on its surface is commonplace. When doing so, however, we consider different choices of p in the r^p-weighted estimates, and this will allow us to avoid the use of time dependent vector fields to obtain the additional decay needed for these terms.

1.1. **Notation.** Here fix the notation that will be used throughout the paper. We let $u' = \partial u = (\partial_t u, \nabla_x u)$ denote the space-time gradient. The notation

$$\Omega = (x_2\partial_3 - x_3\partial_2, x_3\partial_1 - x_1\partial_3, x_1\partial_2 - x_2\partial_1)$$

is used for the generators of rotations. And

$$Z = (\partial_0, \partial_1, \partial_2, \partial_3, \Omega_1, \Omega_2, \Omega_3)$$

will denote our collections of admissible vector fields. We will use the shorthand

$$|Z^{\le N} u| = \sum_{|\mu|\le N} |Z^\mu u|, \quad |\partial^{\le N} u| = \sum_{|\mu|\le N} |\partial^\mu u|.$$

The (spatial) gradient will be frequently (orthogonally) decomposed into its radial and angular parts:

$$\nabla_x = \frac{x}{r}\partial_r + \not\nabla.$$

Here, as is standard, $r = |x|$ and $\partial_r = \frac{x}{r}\cdot\nabla_x$. The components of ∂u that are tangent to the light cone are known to have better decay properties. We will abbreviate these "good" derivatives as

$$\not\partial = (\partial_t + \partial_r, \not\nabla).$$

A key property of the admissible vector fields is that they satisfy:

$$[\Box, Z] = 0.$$

We also need to understand how they interact with ∂ and $\not\partial$. In particular, we have

$$|[Z,\partial]u| \le |\partial u|, \quad |[Z,\not\partial]u| \le \frac{1}{r}|Zu|, \quad |[\partial,\not\partial]u| \le \frac{1}{r}|\partial u|. \tag{1.5}$$

In the second computation, we use the fact that $|\not\nabla u| \le \frac{1}{r}|Zu|$, which follows from $\not\nabla = -\frac{x}{r^2}\times\Omega$.

2. Local Energy Estimates

In order to handle the quasilinear nature of the problem, we will rely on linear estimates for the wave equation on geometries that are a small, though time-dependent, perturbation of Minkowski space. In particular, we consider solutions to

$$\begin{aligned} (\Box_h u)^I &= F^I, \\ u(0,\,\cdot\,) = f, &\quad \partial_t u(0,\,\cdot\,) = g \end{aligned} \tag{2.1}$$

where

$$(\Box_h u)^I = (\partial_t^2 - \Delta)u^I + h^{I,\alpha\beta}_K \partial_\alpha\partial_\beta u^K$$

and the perturbations are assumed to satisfy

$$h^{I,\alpha\beta}_K = h^{K,\alpha\beta}_I = h^{I,\beta\alpha}_K. \tag{2.2}$$

For a differential operator D, we use the following notation

$$|Dh| = \sum_{I,K=1}^{M}\sum_{\alpha,\beta=0}^{3} |Dh^{I,\alpha\beta}_K(t,x)|, \quad |D(h^{\alpha\beta}\omega_\alpha\omega_\beta)| = \sum_{I,K=1}^{M} |D(h^{I,\alpha\beta}_K(t,x)\omega_\alpha\omega_\beta)|.$$

In [15], the integrated local energy estimate was established for $\Box_h$ and used to prove global existence for systems of wave equations satisfying the null condition in exterior domains. We will utilize the notation

$$\|u\|_{LE} = \sup_{R\ge 1} R^{-1/2}\|u\|_{L^2_t L^2_x([0,\infty)\times\{\frac{R}{2}\le\langle x\rangle\le R\})}, \quad \|u\|_{LE^1} = \|(\partial u, u/r)\|_{LE}$$

and record the following immediate corollary of [15, Proposition 2.2].

Proposition 2.1. *Suppose that h satisfies* (2.2) *and*

$$|h| = \sum_{I,K=1}^{M}\sum_{\alpha,\beta=0}^{3} |h^{I,\alpha\beta}_K(t,x)| \le \delta \ll 1 \tag{2.3}$$

with $\delta > 0$ sufficiently small. Then if $u \in C^\infty$ solves (2.1) *and for every t, $|\partial^{\le 1}u(t,x)| \to 0$ as $|x| \to \infty$, then*

$$\|u\|^2_{LE^1} + \|\partial u\|^2_{L^\infty_t L^2_x} \lesssim \|\partial u(0,\,\cdot\,)\|^2_{L^2} + \int_0^\infty\int \Big(|\partial u| + \frac{|u|}{\langle r\rangle}\Big)|\Box_h u|\,dx\,dt \tag{2.4}$$
$$+ \int_0^\infty\int \Big|\partial_\alpha h^{I,\alpha\beta}_K \partial_\beta u^K\Big|\Big(|\partial u^I| + \frac{|u^I|}{r}\Big)\,dx\,dt + \int_0^\infty\int \Big|(\partial h^{I,\alpha\beta}_K)\partial_\beta u^K \partial_\alpha u^I\Big|\,dx\,dt$$
$$+ \int_0^\infty\int \frac{|h|}{\langle r\rangle}|\partial u|\Big(|\partial u| + \frac{|u|}{r}\Big)\,dx\,dt.$$

The spatial portion of the LE^1 norm considers the local energy of u in an inhomogeneous annulus with a weight that is dictated by the radii of the annulus. The estimate captures the fact that this local energy decays at a sufficiently rapid rate to permit L^2-integrability in time with a bound that (essentially) matches that provided by the energy estimate (for perturbations of $\Box$).

The proof of this proposition follows upon pairing $(\Box_h u)^I$ with a multiplier of the form

$$C\partial_t u^I + \frac{r}{r+R}\partial_r u^I + \frac{1}{r+R}u^I,$$

integrating over $[0,T]\times\mathbb{R}^3$, and integrating by parts. See also [17] and [14].

If we set $\omega=(-1,x/r)$, in order to take advantage of the null condition in the sequel, we note

$$\begin{aligned}\partial_\alpha h^{I,\alpha\beta}_K\partial_\beta u^K =&(\partial_\alpha-\omega_\alpha\partial_r)h^{I,\alpha\beta}_K\partial_\beta u^K+\omega_\alpha\partial_r h^{I,\alpha\beta}_K(\partial_\beta-\omega_\beta\partial_r)u^K\\ &+\partial_r(\omega_\alpha\omega_\beta h^{I,\alpha\beta}_K)\partial_r u^K,\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}\partial h^{I,\alpha\beta}_K\partial_\beta u^K\partial_\alpha u^I =&\partial h^{I,\alpha\beta}_K(\partial_\beta-\omega_\beta\partial_r)u^K\partial_\alpha u^I+\partial h^{I,\alpha\beta}\omega_\beta\partial_r u^K(\partial_\alpha-\omega_\alpha\partial_r)u^I\\ &+(\omega_\alpha\omega_\beta\partial h^{I,\alpha\beta})\partial_r u^K\partial_r u^I.\end{aligned}\tag{2.6}$$

Using these, we observe that (2.4) implies that

$$\begin{aligned}\|u\|^2_{LE^1}+\|\partial u\|^2_{L^\infty_tL^2_x}\lesssim{}&\|\partial u(0,\,\cdot\,)\|^2_{L^2}+\int_0^\infty\int\Big(|\partial u|+\frac{|u|}{\langle r\rangle}\Big)|\Box_h u|\,dx\,dt\\ &+\int_0^\infty\int|\partial h||\bar\partial u|\Big(|\partial u|+\frac{|u|}{r}\Big)\,dx\,dt\\ &+\int_0^\infty\int\Big(\frac{|h|}{\langle r\rangle}+|\bar\partial h|+|\partial(\omega_\alpha\omega_\beta h^{\alpha\beta})|\Big)|\partial u|\Big(|\partial u|+\frac{|u|}{r}\Big)\,dx\,dt.\end{aligned}\tag{2.7}$$

We next consider a variant of the r^p-weighted local energy estimate of [2]. To, e.g., readily control commutators involving vector fields and $\bar\partial$, the following estimate that is akin to a Hardy inequality is convenient.

Lemma 2.2. *Suppose that $u\in C^1([0,\infty)\times\mathbb{R}^3)$ and that for each $t\in[0,\infty)$, $r^{p/2}|u(t,x)|\to 0$ as $|x|\to\infty$. Then, provided $0<p<2$,*

$$\|r^{\frac{p-2}{2}}u\|_{L^\infty_tL^2_x}+\|r^{\frac{p-3}{2}}u\|_{L^2_tL^2_x}\lesssim\|r^{\frac{p-2}{2}}u(0,\,\cdot\,)\|_{L^2}+\|r^{\frac{p-3}{2}}\bar\partial(ru)\|_{L^2_tL^2_x}.\tag{2.8}$$

Proof. We consider

$$\int_0^T\int r^{p-3}u^2\,dx\,dt=\frac{1}{p-2}\int_0^T\int_{\mathbb{S}^2}\int_0^\infty(\partial_t+\partial_r)(r^{p-2})(ru)^2\,dr\,d\sigma\,dt.$$

Upon integrating by parts, this is

$$\begin{aligned}&=\frac{1}{p-2}\int r^{p-2}u^2(T,x)\,dx-\frac{1}{p-2}\int r^{p-2}u^2(0,x)\,dx\\ &\quad+\frac{2}{2-p}\int_0^T\int_{\mathbb{S}^2}\int_0^\infty r^{p-2}(ru)(\partial_t+\partial_r)(ru)\,dr\,d\sigma\,dt.\end{aligned}$$

If we apply the Schwarz inequality to the last term, the above two equations yield (where the norms in time are taken over $[0,T]$)

$$\|r^{\frac{p-3}{2}}u\|^2_{L^2_tL^2_x}+\|r^{\frac{p-2}{2}}u(T,\,\cdot\,)\|^2_{L^2_x}\lesssim\|r^{\frac{p-2}{2}}u(0,\,\cdot\,)\|^2_{L^2_x}+\|r^{\frac{p-3}{2}}u\|_{L^2_tL^2_x}\|r^{\frac{p-3}{2}}(\partial_t+\partial_r)(ru)\|_{L^2_tL^2_x}.$$

Using that $ab\le ca^2+\frac{1}{4c}b^2$ for any $c>0$, the first factor of the last term can be absorbed into the left side. The proof is then completed by taking a supremum over T. □

The next result is the main linear estimate used in our proof of global existence. It is based on the r^p-weighted local energy estimates of [2]. Here we have adapted the proof to allow for small, time-dependent perturbations of the geometry in order to accommodate the quasilinear nature of the problem. Due to the "problem of multiple good derivatives," we do so in two different ways. The first estimate, which will be applied with the highest order of vector fields, uses integration by parts on the perturbation in the most standard way. Upon doing so, it is possible that both good derivatives will land on the perturbation. To handle this, the second estimate, which will be used at a lower order and with a higher p, will be employed. In this second case, if neither of the derivatives in the quasilinear term are good derivatives, no further integration by parts will be applied. This will keep the two good derivatives on separate terms, which will each be at this lower order.

Theorem 2.3. *Suppose $h\in C^2([0,\infty)\times\mathbb{R}^3)$ satisfies* (2.2)*. Let $u\in C^2([0,\infty)\times\mathbb{R}^3)$ be so that for each $t\ge 0$,*

$$r^{\frac{p+2}{2}}|\partial^{\le 1}u(t,x)|\to 0\ \text{as}\ |x|\to\infty.$$

Then, for any $0<p<2$,

(2.9)

$$\begin{aligned}
&\|r^{\frac{p-1}{2}}\bar\partial u\|^2_{L^2_tL^2_x}+\|r^{\frac{p-3}{2}}u\|^2_{L^2_tL^2_x}+\|r^{\frac{p}{2}}\bar\partial u\|^2_{L^\infty_tL^2_x}+\|r^{\frac{p-2}{2}}u\|^2_{L^\infty_tL^2_x}\\
&\quad\lesssim\|r^{\frac{p}{2}}\bar\partial u(0,\,\cdot\,)\|^2_{L^2_x}+\|r^{\frac{p-2}{2}}u(0,\,\cdot\,)\|^2_{L^2_x}\\
&\quad+\sup_t\Big(\int r^p|h||\partial u|\Big(|\partial u|+r^{-1}|u|\Big)\,dx\Big)\\
&\quad+\int_0^\infty\int r^p|\Box_h u|\Big(|\bar\partial u|+\frac{|u|}{r}\Big)\,dx\,dt+\int_0^\infty\int r^p|\partial h||\bar\partial u|\Big(|\bar\partial u|+\frac{|u|}{r}\Big)\,dx\,dt\\
&\quad+\int_0^\infty\int r^p\Big(\frac{|h|}{r}+|\bar\partial h|+|\partial(\omega_\alpha\omega_\beta h^{\alpha\beta})|\Big)\ |\partial u|\Big(|\partial u|+\frac{|u|}{r}\Big)\,dx\,dt,
\end{aligned}$$

and

(2.10)

$$\|r^{\frac{p-1}{2}}\bar\partial u\|^2_{L^2_tL^2_x}+\|r^{\frac{p-3}{2}}u\|^2_{L^2_tL^2_x}+\|r^{\frac{p}{2}}\bar\partial u\|^2_{L^\infty_tL^2_x}+\|r^{\frac{p-2}{2}}u\|^2_{L^\infty_tL^2_x}$$
$$\lesssim \|r^{\frac{p}{2}}\bar\partial u(0,\,\cdot)\|^2_{L^2_x}+\|r^{\frac{p-2}{2}}u(0,\,\cdot)\|^2_{L^2_x}$$
$$+\sup_t\Big(\int r^p|h|\Big(|\partial u|+\frac{|u|}{r}\Big)\Big(|\bar\partial u|+\frac{|u|}{r}\Big)\,dx\Big)$$
$$+\int_0^\infty\int r^p|\Box_h u|\Big(|\bar\partial u|+\frac{|u|}{r}\Big)\,dx\,dt$$
$$+\int_0^\infty\int r^{p-1}|h||\partial u|\Big(|\bar\partial u|+\frac{|u|}{r}\Big)\,dx\,dt+\int_0^\infty\int r^p|\partial h||\bar\partial u|\Big(|\bar\partial u|+\frac{|u|}{r}\Big)\,dx\,dt$$
$$+\int_0^\infty\int r^p\Big(|\bar\partial h|+|h^{\alpha\beta}\omega_\alpha\omega_\beta|\Big)|\partial\partial^{\le 1}u|\Big(|\bar\partial u|+\frac{|u|}{r}\Big)\,dx\,dt$$
$$+\int_0^\infty\int r^{p-3}\Big(r|\partial h|+|h|\Big)|u|^2.$$

The reader should have in mind that in the sequel we will choose the perturbation to have the form $h^{I,\alpha\beta}_K=-B^{I,\gamma\alpha\beta}_{JK}\partial_\gamma u^J$. We note that the last term in (2.9) could potentially have multiple good derivatives when the perturbation h is itself based in a good derivative. As indicated above, to remedy this, we will later use (2.10) with its p chosen to be more than twice that used in (2.9). We also point out that the second to last term of (2.10) has a factor containing more derivatives than what appear in the left, which requires that this estimate be applied with a lower number of vector fields so this loss of regularity can be overcome.

Proof. We consider

$$\int_0^T\int r^p\Box_h u^I\Big(\partial_t+\partial_r+\frac{1}{r}\Big)u^I\,dx\,dt.$$

To start, we argue as in [3] and note that

$$\int_0^T\int r^p\Box u^I\Big(\partial_t+\partial_r+\frac{1}{r}\Big)u^I\,dx\,dt$$
$$=\int_0^T\int r^p\Big[\Big(\partial_t-\partial_r\Big)\Big(\partial_t+\partial_r\Big)(ru^I)-\not\nabla\cdot\not\nabla(ru^I)\Big]\Big(\partial_t+\partial_r\Big)(ru^I)\,dr\,d\sigma\,dt.$$

Integrating by parts and using $[\nabla,\partial_r]=\frac{1}{r}\not\nabla$, we see that the right side is

$$=\frac12\int_0^T\int r^p\Big(\partial_t-\partial_r\Big)\Big|\Big(\partial_t+\partial_r\Big)(ru)\Big|^2\,dr\,d\sigma\,dt$$

$$+ \frac{1}{2}\int_0^T \int r^p \Big(\partial_t + \partial_r\Big)|\nabla\!\!\!/ (ru)|^2 \,dr\,d\sigma\,dt$$
$$+ \int_0^T \int r^{p-1}|\nabla\!\!\!/ (ru)|^2 \,dr\,d\sigma\,dt.$$

Subsequent integrations by parts give that this is

$$= \frac{1}{2}\|r^{\frac{p-2}{2}}\partial\!\!\!/ (ru)(T,\,\cdot)\|_{L^2}^2 - \frac{1}{2}\|r^{\frac{p-2}{2}}\partial\!\!\!/ (ru)(0,\,\cdot)\|_{L^2}^2$$
$$+ \frac{p}{2}\int_0^T \int r^{p-1}\Big|\Big(\partial_t + \partial_r\Big)(ru)\Big|^2 \,dr\,d\sigma\,dt$$
$$+ \Big(1 - \frac{p}{2}\Big)\int_0^T \int r^{p-1}|\nabla\!\!\!/ (ru)|^2 \,dr\,d\sigma\,dt.$$

Provided that $0 < p < 2$, we can combine this with (2.8) to obtain

$$\begin{aligned}(2.11)\quad &\|r^{\frac{p-1}{2}}\partial\!\!\!/ u\|^2_{L^2_tL^2_x} + \|r^{\frac{p-3}{2}}u\|^2_{L^2_tL^2_x} + \|r^{\frac{p}{2}}\partial\!\!\!/ u(T,\,\cdot)\|^2_{L^2_x} + \|r^{\frac{p-2}{2}}u(T,\,\cdot)\|^2_{L^2_x}\\ &\lesssim \|r^{\frac{p}{2}}\partial\!\!\!/ u(0,\,\cdot)\|^2_{L^2_x} + \|r^{\frac{p-2}{2}}u(0,\,\cdot)\|^2_{L^2_x} + \Big|\int_0^T \int r^p \Box u^I \Big(\partial_t + \partial_r + \frac{1}{r}\Big)u^I \,dx\,dt\Big|.\end{aligned}$$

We now consider the perturbation terms. Using, again, that $[\nabla, \partial_r] = \frac{1}{r}\nabla\!\!\!/$ and the symmetries (2.2), we obtain

$$\int_0^T \int r^p h^{I,\alpha\beta}_K \partial_\alpha\partial_\beta u^K \Big(\partial_t + \partial_r + \frac{1}{r}\Big)u^I \,dx\,dt$$
$$= \int r^p h^{I,0\beta}_K \partial_\beta u^K \Big(\partial_t + \partial_r + \frac{1}{r}\Big)u^I \,dx\Big|_{t=0}^T$$
$$- \int_0^T \int r^{p-1} h^{I,j\beta}_K \partial_\beta u^K \nabla\!\!\!/_j u^I \,dx\,dt$$
$$+ \int_0^T \int r^{p-2}\omega_j h^{I,j\beta}_K \partial_\beta u^K u^I \,dx\,dt$$
$$- \frac{1}{2}\int_0^T \int r^p h^{I,\alpha\beta}_K \Big(\partial_t + \partial_r + \frac{1}{r}\Big)\Big[\partial_\beta u^K \partial_\alpha u^I\Big] \,dx\,dt$$
$$- \int_0^T \int r^p \partial_\alpha h^{I,\alpha\beta}_K \partial_\beta u^K \Big(\partial_t + \partial_r + \frac{1}{r}\Big)u^I \,dx\,dt$$
$$- p\int_0^T \int r^{p-1}\omega_j h^{I,j\beta}_K \partial_\beta u^K \Big(\partial_t + \partial_r + \frac{1}{r}\Big)u^I \,dx\,dt,$$

where, as above, we have set $\omega = (-1, x/r)$. And thus, arguing as in (2.5) and (2.6) and integrating by parts, this is

$$\begin{aligned}
&= \int r^p h^{I,0\beta}_K \partial_\beta u^K \Big(\partial_t + \partial_r + \frac{1}{r}\Big) u^I\,dx\Big|_{t=0}^T - \frac{1}{2}\int r^p h^{I,\alpha\beta}_K \partial_\beta u^K \partial_\alpha u^I\,dx\Big|_{t=0}^T \\
&\quad - \int_0^T \int r^{p-1} h^{I,j\beta}_K \partial_\beta u^K \not\nabla_j u^I\,dx\,dt + \int_0^T \int r^{p-2} h^{I,j\beta}_K u^I \partial_\beta u^K\,dx\,dt \\
&\quad + \frac{p+1}{2}\int_0^T \int r^{p-1} h^{I,\alpha\beta}_K \partial_\beta u^K \partial_\alpha u^I\,dx\,dt \\
&\quad + \frac{1}{2}\int_0^T \int r^p \Big(\partial_t + \partial_r\Big) h^{I,\alpha\beta}_K \partial_\beta u^K \partial_\alpha u^I\,dx\,dt \\
&\quad - \int_0^T \int r^p \partial_\alpha h^{I,\alpha\beta}_K (\partial_\beta - \omega_\beta\partial_r) u^K \Big(\partial_t + \partial_r + \frac{1}{r}\Big) u^I\,dx\,dt \\
&\quad - \int_0^T \int r^p \omega_\beta \Big(\partial_\alpha - \omega_\alpha\partial_r\Big) h^{I,\alpha\beta}_K \partial_r u^K \Big(\partial_t + \partial_r + \frac{1}{r}\Big) u^I\,dx\,dt \\
&\quad - \int_0^T \int r^p \omega_\beta\omega_\alpha \partial_r h^{I,\alpha\beta}_K \partial_r u^K \Big(\partial_t + \partial_r + \frac{1}{r}\Big) u^I\,dx\,dt \\
&\quad - p\int_0^T \int r^{p-1} \omega_j h^{I,j\beta}_K \partial_\beta u^K \Big(\partial_t + \partial_r + \frac{1}{r}\Big) u^I\,dx\,dt.
\end{aligned}$$

From this, when combined with (2.11), the bound (2.9) follows immediately.

We next consider (2.10). We write

$$\begin{aligned}
h^{I,\alpha\beta}_K \partial_\alpha\partial_\beta u^K =& h^{I,\alpha\beta}_K \omega_\alpha\omega_\beta \partial_r^2 u^K + h^{I,\alpha\beta}_K \partial_\alpha \Big(\partial_\beta - \omega_\beta\partial_r\Big) u^K \\
&+ h^{I,\alpha\beta}_K \Big(\partial_\alpha - \omega_\alpha\partial_r\Big)\omega_\beta\partial_r u^K.
\end{aligned}$$

For (2.10), we need not further modify the terms involving $h^{I,\alpha\beta}_K \omega_\alpha\omega_\beta\partial_r^2 u^K$. For those that remain, we notice that

$$\begin{aligned}
&\int_0^T \int r^p h^{I,\alpha\beta}_K \Big(\partial_\alpha\partial_\beta - \omega_\alpha\omega_\beta\partial_r^2\Big) u^K \Big(\partial_t + \partial_r + \frac{1}{r}\Big) u^I\,dx\,dt \\
&\quad = \int r^p h^{I,0\beta}_K \partial_\beta u^K \Big(\partial_t + \partial_r + \frac{1}{r}\Big) u^I\,dx\Big|_{t=0}^T \\
&\quad - p\int_0^T \int r^{p-1} \omega_j h^{I,j\beta,}_K \partial_\beta u^K \Big(\partial_t + \partial_r + \frac{1}{r}\Big) u^I\,dx\,dt \\
&\quad - \int_0^T \int r^p \partial_\alpha h^{I,\alpha\beta}_K \Big(\partial_\beta - \omega_\beta\partial_r\Big) u^K \Big(\partial_t + \partial_r + \frac{1}{r}\Big) u^I\,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^T \int r^p (\partial_\alpha - \omega_\alpha \partial_r) h^{I,\alpha\beta}_K \omega_\beta \partial_r u^K \Big(\partial_t + \partial_r + \frac{1}{r}\Big) u^I \, dx\, dt \\
&+ (p+2) \int_0^T \int r^{p-1} h^{I,\alpha\beta}_K \omega_\alpha \omega_\beta \partial_r u^K \Big(\partial_t + \partial_r + \frac{1}{r}\Big) u^I \, dx\, dt \\
&- \int_0^T \int r^{p-1} h^{I,j\beta}_K \partial_\beta u^K \nabla\!\!\!/_j u^I \, dx\, dt + \int_0^T \int r^{p-2} h^{I,j\beta}_K \omega_j u^I \partial_\beta u^K \, dx\, dt \\
&- \int_0^T \int r^{p-2} h^{I,\alpha\beta}_K \omega_\alpha \omega_\beta u^I \partial_r u^K \, dx\, dt \\
&- \frac{1}{2} \int_0^T \int r^p h^{I,\alpha\beta}_K \Big(\partial_t + \partial_r + \frac{1}{r}\Big) \Big[(\partial_\beta - \omega_\beta \partial_r) u^K \partial_\alpha u^I \\
&\qquad\qquad + \omega_\beta \partial_r u^K (\partial_\alpha - \omega_\alpha \partial_r) u^I \Big] \, dx\, dt.
\end{aligned}
$$

A subsequent integration by parts gives that

$$
\begin{aligned}
&- \frac{1}{2} \int_0^T \int r^p h^{I,\alpha\beta}_K \Big(\partial_t + \partial_r + \frac{1}{r}\Big) \Big[(\partial_\beta - \beta \partial_r) u^K \partial_\alpha u^I \\
&\qquad\qquad + \omega_\beta \partial_r u^K (\partial_\alpha - \omega_\alpha \partial_r) u^I \Big] \, dx\, dt \\
&= -\frac{1}{2} \int r^p h^{I,\alpha\beta}_K \Big[(\partial_\beta - \omega_\beta \partial_r) u^K \partial_\alpha u^I \\
&\qquad\qquad + \omega_\beta \partial_r u^K (\partial_\alpha - \omega_\alpha \partial_r) u^I \Big] \, dx \Big|_0^T \\
&\quad + \frac{1}{2} \int_0^T \int r^p (\partial_t + \partial_r) h^{I,\alpha\beta}_K \Big[(\partial_\beta - \omega_\beta \partial_r) u^K \partial_\alpha u^I \\
&\qquad\qquad + \omega_\beta \partial_r u^K (\partial_\alpha - \omega_\alpha \partial_r) u^I \Big] \, dx\, dt \\
&\quad + \frac{p+1}{2} \int_0^T \int r^{p-1} h^{I,\alpha\beta}_K \Big[(\partial_\beta - \omega_\beta \partial_r) u^K \partial_\alpha u^I \\
&\qquad\qquad + \omega_\beta \partial_r u^K (\partial_\alpha - \omega_\alpha \partial_r) u^I \Big] \, dx\, dt.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\int_0^T \int r^{p-2} h^{I,j\beta}_K \omega_j u^I \partial_\beta u^K \, dx\, dt - \int_0^T \int r^{p-2} h^{I,\alpha\beta}_K \omega_\alpha \omega_\beta u^I \partial_r u^K \, dx\, dt \\
&\quad = \frac{1}{2} \int r^{p-2} h^{I,j0}_K u^K u^I \, dx \Big|_0^T \\
&\quad - \frac{p-1}{2} \int_0^T \int r^{p-3} \omega_j \omega_l h^{I,jl}_K u^K u^I \, dx\, dt - \frac{1}{2} \int_0^T \int r^{p-3} h^{I,jj}_K u^K u^I \, dx\, dt
\end{aligned}
$$

$$- \frac{1}{2}\int_0^T \int r^{p-2}\omega_j \partial_\beta h^{I,j\beta}_K u^K u^I \,dx\,dt + \frac{p}{2}\int_0^T \int r^{p-3} h^{I,\alpha\beta}_K \omega_\alpha\omega_\beta u^K u^I \,dx\,dt$$
$$+ \frac{1}{2}\int_0^T \int r^{p-2}\partial_r h^{I,\alpha\beta}_K \omega_\alpha\omega_\beta u^K u^I \,dx\,dt.$$

Combining (2.11) with the preceding three equations yields (2.10). □

The main source of decay that we rely upon is the following weighted Sobolev estimate that originates from [8]. It is proved by localizing and applying standard Sobolev embeddings in the r, ω variables. The decay results upon adjusting the volume element $dr\, d\sigma(\omega)$ to match that of $\mathbb{R}^3$ in spherical coordinates.

Lemma 2.4. *For $h \in C^\infty(\mathbb{R}^3)$ and $R \geq 1$, we have*

$$\|h\|_{L^\infty(R/2<|x|<R)} \lesssim R^{-1}\|Z^{\leq 2}h\|_{L^2(R/4<|x|<2R)}. \tag{2.12}$$

We now use the smallness of h to absorb some perturbative terms and first undertake (2.9). The following proposition largely addresses the problem of multiple good derivatives. In the sequel, the perturbation will be a lower order term. When this lower order term is small in a weighted space with the $\tilde{p}$ more than twice the choice of p for the higher order factors, we are able to absorb the perturbative factors, including the last term in (2.9), which is the possible occurrence of multiple good derivatives. The resulting estimate is then quite similar to that used in [3] for the semilinear case. The issue with multiple good derivatives barely appears in the next section as it is entirely reduced to demonstrating hypothesis (2.13) below.

Proposition 2.5. *Fix $0 < p < 1$. Assume that $h \in C^2([0,\infty)\times\mathbb{R}^3)$ satisfies (2.2). Moreover, for $\tilde{p} > 2p$, suppose*

$$\|Z^{\leq 3}h\|_{L^\infty_t L^2_x} + \|\langle r\rangle^{\frac{\tilde{p}-1}{2}} Z^{\leq 2}\bar{\partial} h\|_{L^2_t L^2_x} + \|\langle r\rangle^{\frac{\tilde{p}-1}{2}} Z^{\leq 3}(\omega_\alpha\omega_\beta h^{\alpha\beta})\|_{L^2_t L^2_x} \leq \delta \tag{2.13}$$

for $\delta > 0$ sufficiently small. Let $u \in C^2([0,\infty)\times\mathbb{R}^3)$ be so that for each $t \geq 0$,

$$r^{\frac{p+2}{2}}|\partial^{\leq 1}Z^{\leq N}u(t,x)| \to 0, \qquad \text{as } |x|\to\infty.$$

Then

$$\begin{aligned}\|\langle r\rangle^{\frac{p}{2}}\bar{\partial} Z^{\leq N}u\|_{L^\infty_t L^2_x} &+ \|\langle r\rangle^{\frac{p-2}{2}} Z^{\leq N}u\|_{L^\infty_t L^2_x} + \|\partial Z^{\leq N}u\|_{L^\infty_t L^2_x} \\ &+ \|\langle r\rangle^{\frac{p-1}{2}}\bar{\partial} Z^{\leq N}u\|_{L^2_t L^2_x} + \|\langle r\rangle^{\frac{p-3}{2}} Z^{\leq N}u\|_{L^2_t L^2_x} + \|Z^{\leq N}u\|_{LE^1} \\ &\lesssim \|\langle r\rangle^{\frac{p}{2}}\bar{\partial} Z^{\leq N}u(0,\cdot\,)\|_{L^2_x} + \|\langle r\rangle^{\frac{p-2}{2}} Z^{\leq N}u(0,\cdot\,)\|_{L^2_x} \\ &\qquad + \|\partial Z^{\leq N}u(0,\cdot\,)\|_{L^2_x} + \|\langle r\rangle^{\frac{p+1}{2}}\Box_h Z^{\leq N}u\|_{L^2_t L^2_x}.\end{aligned} \tag{2.14}$$

Proof. We first apply (2.9) to $\chi_{>1}(|x|)Z^{\leq N}u$ where $\chi_{>1}(r)$ is a smooth function that vanishes for $r \leq 1$ and is identically 1 for $r > 2$. Using a Sobolev embedding, $H^2(\mathbb{R}^3) \subseteq L^\infty(\mathbb{R}^3)$, and the bound for the first term in (2.13), it follows that

$$\int_0^\infty \int r^p |[\Box_h, \chi_{>1}]Z^{\leq N}u||\bar{\partial}(\chi_{>1}Z^{\leq N}u)|\,dx\,dt \lesssim (1 + \|\partial^{\leq 1}h\|_{L^\infty_t L^\infty_x})\|Z^{\leq N}u\|^2_{LE^1} \lesssim \|Z^{\leq N}u\|^2_{LE^1}.$$

Thus by subsequently applying (2.7) to $Z^{\leq N}u$, we see that the square of the left side of (2.14) is bounded by

$$\begin{aligned}(2.15)\quad &\|\langle r\rangle^{\frac{p}{2}}\bar{\partial}Z^{\leq N}u(0,\,\cdot\,)\|^2_{L^2_x} + \|\langle r\rangle^{\frac{p-2}{2}}Z^{\leq N}u(0,\,\cdot\,)\|^2_{L^2_x} + \|\partial Z^{\leq N}u(0,\,\cdot\,)\|^2_{L^2_x}\\ &+ \sup_t \int \langle r\rangle^p |h|\Big(|\partial Z^{\leq N}u| + \frac{|Z^{\leq N}u|}{\langle r\rangle}\Big)^2\,dx\\ &+ \int_0^\infty\int \Big(\langle r\rangle^p|\bar{\partial}Z^{\leq N}u| + |\partial Z^{\leq N}u| + \langle r\rangle^{p-1}|Z^{\leq N}u|\Big)|\Box_h Z^{\leq N}u|\,dx\,dt\\ &+ \int_0^\infty\int \langle r\rangle^p|\partial h|\Big(|\bar{\partial}Z^{\leq N}u| + \frac{|Z^{\leq N}u|}{\langle r\rangle}\Big)^2\,dx\,dt\\ &+ \int_0^\infty\int |\partial h||\bar{\partial}Z^{\leq N}u|\Big(|\partial Z^{\leq N}u| + \frac{|Z^{\leq N}u|}{r}\Big)\,dx\,dt\\ &+ \int_0^\infty\int \langle r\rangle^p\Big(\frac{|h|}{\langle r\rangle} + |\bar{\partial}h| + |\partial(\omega_\alpha\omega_\beta h^{\alpha\beta})|\Big)\Big(|\partial Z^{\leq N}u| + \frac{|Z^{\leq N}u|}{r}\Big)^2\,dx\,dt.\end{aligned}$$

By the Cauchy-Schwarz inequality and the fact that $p > 0$, we obtain

$$\begin{aligned}&\int_0^\infty\int \Big(\langle r\rangle^p|\bar{\partial}Z^{\leq N}u| + |\partial Z^{\leq N}u| + \langle r\rangle^{p-1}|Z^{\leq N}u|\Big)|\Box_h Z^{\leq N}u|\,dx\,dt\\ &\quad\leq \frac{1}{2}\Big(\|\langle r\rangle^{\frac{p-1}{2}}\bar{\partial}Z^{\leq N}u\|^2_{L^2_tL^2_x} + \|Z^{\leq N}u\|^2_{LE^1} + \|\langle r\rangle^{\frac{p-3}{2}}Z^{\leq N}u\|^2_{L^2_tL^2_x}\Big)\\ &\qquad + C\|\langle r\rangle^{\frac{p+1}{2}}\Box_h Z^{\leq N}u\|^2_{L^2_tL^2_x}.\end{aligned}$$

The first three terms in the right side can be absorbed by the square of the left side of (2.14).

We will proceed to showing that the fourth, sixth, seventh, and eighth terms of (2.15) can be bounded by a constant that can be chosen sufficiently small times the square of the left side of (2.14). These terms can again be absorbed, which will complete the argument.

Since $p < 1$, using (2.12), a standard Hardy inequality

$$(2.16)\qquad \|r^{-1}u\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla u\|_{L^2(\mathbb{R}^3)},$$

and (2.13) results in

$$\sup_t \int \langle r\rangle^p |h| \Big(|\partial Z^{\le N} u| + \frac{|Z^{\le N}u|}{\langle r\rangle}\Big)^2\,dx \lesssim \delta \|\partial Z^{\le N} u\|^2_{L^\infty_t L^2_x},$$

which suffices for the fourth term in (2.15).

Proceeding to the sixth term in (2.15), we apply (2.12) and (2.13) to get

$$\begin{aligned}\int_0^\infty \int \langle r\rangle^p |\partial h| \Big(|\bar\partial Z^{\le N} u| + \frac{|Z^{\le N}u|}{\langle r\rangle}\Big)^2\,dx\,dt \\ \lesssim \delta\Big(\|\langle r\rangle^{\frac{p-1}{2}} \bar\partial Z^{\le N} u\|^2_{L^2_t L^2_x} + \|\langle r\rangle^{\frac{p-3}{2}} Z^{\le N}u\|^2_{L^2_tL^2_x}\Big).\end{aligned}$$

Similarly, since $p>0$,

$$\begin{aligned}\int_0^\infty \int |\partial h||\bar\partial Z^{\le N} u| \Big(|\partial Z^{\le N} u| + \frac{|Z^{\le N}u|}{r}\Big)\,dx\,dt \\ \lesssim \delta \|\langle r\rangle^{\frac{p-1}{2}} \bar\partial Z^{\le N} u\|_{L^2_t L^2_x} \|Z^{\le N}u\|_{LE^1}.\end{aligned}$$

And since $p<1$, (2.12) and (2.13) give

$$\int_0^\infty \int \langle r\rangle^{p-1} |h| \Big(|\partial Z^{\le N} u| + \frac{|Z^{\le N}u|}{\langle r\rangle}\Big)^2\,dx\,dt \lesssim \delta\|Z^{\le N}u\|^2_{LE^1}.$$

It remains to establish

$$\begin{aligned}\int_0^\infty \int \langle r\rangle^p \Big(|\bar\partial h| + |\partial(\omega_\alpha\omega_\beta h^{\alpha\beta})|\Big)\Big(|\partial Z^{\le N} u| + \frac{|Z^{\le N}u|}{r}\Big)^2\,dx\,dt \\ \lesssim \delta \|\partial Z^{\le N} u\|_{L^\infty_t L^2_x} \|Z^{\le N}u\|_{LE^1}.\end{aligned} \tag{2.17}$$

The left side of (2.17) is bounded by

$$\sum_{j\ge 0} 2^{pj} \int_0^\infty \int_{\{2^{j-1}\le \langle x\rangle \le 2^j\}} \Big(|\bar\partial h| + |\partial(\omega_\alpha\omega_\beta h^{\alpha\beta})|\Big)\Big(|\partial Z^{\le N} u| + \frac{|Z^{\le N}u|}{r}\Big)^2\,dx\,dt.$$

Applying (2.12), the Schwarz inequality, and the Hardy inequality (2.16), this is controlled by

$$\begin{aligned}\Big(\sum_{j\ge 0} 2^{j(p-\frac{\tilde p}{2})}\Big)\Big(\|\langle r\rangle^{\frac{\tilde p -1}{2}} Z^{\le 2}\bar\partial h\|_{L^2_tL^2_x} + \|\langle r\rangle^{\frac{\tilde p -1}{2}} Z^{\le 2}\partial(\omega_\alpha\omega_\beta h^{\alpha\beta})\|_{L^2_tL^2_x}\Big) \\ \cdot \|\partial Z^{\le N} u\|_{L^\infty_t L^2_x} \|Z^{\le N}u\|_{LE^1}.\end{aligned}$$

Using the bound on the last two terms of (2.13), as $\tilde p > 2p$, (2.17) follows. □

We next consider a result analogous to Proposition 2.5 for the lower order energy, which has the larger weight $\tilde p$. The proof proceeds similarly but is based instead on (2.10). It is this estimate that will allow us to show (2.13) in the sequel, which thus addresses the issue with multiple good derivatives. We note that the following will be applied at a lower order than (2.14), and as such, we will be able to handle the loss of a vector field that occurs in the last two terms of the estimate.

Proposition 2.6. *Fix $0 < p < 1$. Assume that $h \in C^2([0,\infty)\times\mathbb{R}^3)$ satisfies (2.2) and, for $2p < \tilde{p} < 2$, (2.13) with $\delta > 0$ sufficiently small. Let $u \in C^2([0,\infty)\times\mathbb{R}^3)$ be so that for each $t \geq 0$,*

$$r^{\frac{p+2}{2}}|\partial^{\leq 1} Z^{\leq N} u(t,x)| \to 0, \qquad \text{as } |x| \to \infty.$$

Then

$$\begin{aligned}(2.18)\quad &\|\langle r\rangle^{\frac{\tilde{p}}{2}} \bar{\partial} Z^{\leq N-1} u\|_{L^\infty_t L^2_x} + \|\langle r\rangle^{\frac{\tilde{p}-2}{2}} Z^{\leq N-1} u\|_{L^\infty_t L^2_x} \\ &\quad + \|\langle r\rangle^{\frac{\tilde{p}-1}{2}} \bar{\partial} Z^{\leq N-1} u\|_{L^2_t L^2_x} + \|\langle r\rangle^{\frac{\tilde{p}-3}{2}} Z^{\leq N-1} u\|_{L^2_t L^2_x} \\ &\lesssim \|\langle r\rangle^{\frac{\tilde{p}}{2}} \bar{\partial} Z^{\leq N-1} u(0,\,\cdot\,)\|_{L^2_x} + \|\langle r\rangle^{\frac{\tilde{p}-2}{2}} Z^{\leq N-1} u(0,\,\cdot\,)\|_{L^2_x} \\ &\quad + \|\langle r\rangle^{\frac{\tilde{p}+1}{2}} \Box_h Z^{\leq N-1} u\|_{L^2_t L^2_x} + \|Z^{\leq N} u\|_{LE^1} + \|\partial Z^{\leq N} u\|_{L^\infty_t L^2_x}.\end{aligned}$$

Proof. As in the preceding proof, (2.10) can be applied to control the square of the left side of (2.18) by

$$\begin{aligned}&\|\langle r\rangle^{\frac{\tilde{p}}{2}} \bar{\partial} Z^{\leq N-1} u(0,\,\cdot\,)\|^2_{L^2_x} + \|\langle r\rangle^{\frac{\tilde{p}-2}{2}} Z^{\leq N-1} u(0,\,\cdot\,)\|^2_{L^2_x} \\ &\quad + \sup_t \Big(\int \langle r\rangle^{\tilde{p}} |h| \Big(|\partial Z^{\leq N-1} u| + \frac{|Z^{\leq N-1} u|}{\langle r\rangle}\Big)\Big(|\bar{\partial} Z^{\leq N-1} u| + \frac{|Z^{\leq N-1} u|}{\langle r\rangle}\Big)\,dx\Big) \\ &\qquad + \int_0^\infty \int \langle r\rangle^{\tilde{p}} |\Box_h Z^{\leq N-1} u| \Big(|\bar{\partial} Z^{\leq N-1} u| + \frac{|Z^{\leq N-1} u|}{\langle r\rangle}\Big)\,dx\,dt \\ &\qquad + \int_0^\infty \int \langle r\rangle^{\tilde{p}-1} |h| |\partial Z^{\leq N-1} u| \Big(|\bar{\partial} Z^{\leq N-1} u| + \frac{|Z^{\leq N-1} u|}{\langle r\rangle}\Big)\,dx\,dt \\ &\qquad + \int_0^\infty \int \langle r\rangle^{\tilde{p}} |\partial h| |\bar{\partial} Z^{\leq N-1} u| \Big(|\bar{\partial} Z^{\leq N-1} u| + \frac{|Z^{\leq N-1} u|}{\langle r\rangle}\Big)\,dx\,dt \\ &\quad + \int_0^\infty \int \langle r\rangle^{\tilde{p}} \Big(|\bar{\partial} h| + |h^{\alpha\beta}\omega_\alpha\omega_\beta|\Big) |\partial Z^{\leq N} u| \Big(|\bar{\partial} Z^{\leq N-1} u| + \frac{|Z^{\leq N-1} u|}{\langle r\rangle}\Big)\,dx\,dt \\ &+ \int_0^\infty \int \langle r\rangle^{\tilde{p}-3} \Big(r|\partial h| + |h|\Big) |Z^{\leq N-1} u|^2\,dx\,dt + \|Z^{\leq N-1} u\|^2_{LE^1} + \|\partial Z^{\leq N-1} u\|^2_{L^\infty_t L^2_x}.\end{aligned}$$

Using the Schwarz inequality, we see that

$$\int_0^\infty \int \langle r\rangle^{\tilde{p}} |\Box_h Z^{\leq N-1} u| \Big(|\bar{\partial} Z^{\leq N-1} u| + \frac{|Z^{\leq N-1} u|}{\langle r\rangle}\Big)\,dx\,dt$$

$$\le \|\langle r\rangle^{\frac{\tilde p+1}{2}}\Box_h Z^{\le N-1}u\|_{L^2_tL^2_x}\Big(\|\langle r\rangle^{\frac{\tilde p-1}{2}}\bar\partial Z^{\le N-1}u\|_{L^2_tL^2_x}+\|\langle r\rangle^{\frac{\tilde p-3}{2}}Z^{\le N-1}u\|_{L^2_tL^2_x}\Big)$$

and the second factor can be absorbed by the square of the left side of (2.18) after applying $ab\le ca^2+\frac{1}{4c}b^2$.

As above, we now seek to control the third, fifth, sixth, seventh, and eighth terms by a small parameter times the square of the left side of (2.18). These terms can then be absorbed for a sufficiently small choice of the parameter, which will complete the proof.

We first note that

$$\begin{aligned}\sup_t\int \langle r\rangle^{\tilde p}|h|\Big(|\partial Z^{\le N-1}u|+\frac{|Z^{\le N-1}u|}{\langle r\rangle}\Big)\Big(|\bar\partial Z^{\le N-1}u|+\frac{|Z^{\le N-1}u|}{\langle r\rangle}\Big)\,dx\\ \lesssim \|Z^{\le 2}h\|_{L^\infty_tL^2_x}\|\partial Z^{\le N-1}u\|_{L^\infty_tL^2_x}\\ \cdot\Big(\|\langle r\rangle^{\frac{\tilde p}{2}}\bar\partial Z^{\le N-1}u\|_{L^\infty_tL^2_x}+\|\langle r\rangle^{\frac{\tilde p-2}{2}}Z^{\le N-1}u\|_{L^\infty_tL^2_x}\Big)\\ \lesssim \delta\Big(\|\partial Z^{\le N-1}u\|^2_{L^\infty_tL^2_x}+\|\langle r\rangle^{\frac{\tilde p}{2}}\bar\partial Z^{\le N-1}u\|^2_{L^\infty_tL^2_x}+\|\langle r\rangle^{\frac{\tilde p-2}{2}}Z^{\le N-1}u\|^2_{L^\infty_tL^2_x}\Big)\end{aligned}$$

where we have used (2.12), (2.16), (2.13), and the assumption that $\tilde p<2$.

For the remaining terms, we repeatedly use the hypothesis $\tilde p<2$, (2.12), and (2.13) and obtain the bounds:

$$\begin{aligned}\int_0^\infty\int \langle r\rangle^{\tilde p-1}|h||\partial Z^{\le N-1}u|\Big(|\bar\partial Z^{\le N-1}u|+\frac{|Z^{\le N-1}u|}{\langle r\rangle}\Big)\,dx\,dt\\ \lesssim \delta\|Z^{\le N-1}u\|_{LE^1}\Big(\|\langle r\rangle^{\frac{\tilde p-1}{2}}\bar\partial Z^{\le N-1}u\|_{L^2_tL^2_x}+\|\langle r\rangle^{\frac{\tilde p-3}{2}}Z^{\le N-1}u\|_{L^2_tL^2_x}\Big),\end{aligned}$$

$$\begin{aligned}\int_0^\infty\int \langle r\rangle^{\tilde p}|\partial h||\bar\partial Z^{\le N-1}u|\Big(|\bar\partial Z^{\le N-1}u|+\frac{|Z^{\le N-1}u|}{\langle r\rangle}\Big)\,dx\,dt\\ \lesssim \delta\|\langle r\rangle^{\frac{\tilde p-1}{2}}\bar\partial Z^{\le N-1}u\|_{L^2_tL^2_x}\Big(\|\langle r\rangle^{\frac{\tilde p-1}{2}}\bar\partial Z^{\le N-1}u\|_{L^2_tL^2_x}+\|\langle r\rangle^{\frac{\tilde p-3}{2}}Z^{\le N-1}u\|_{L^2_tL^2_x}\Big),\end{aligned}$$

and

$$\int_0^\infty\int \langle r\rangle^{\tilde p-3}\Big(r|\partial h|+|h|\Big)|Z^{\le N-1}u|^2\,dx\,dt\lesssim \delta\|\langle r\rangle^{\frac{\tilde p-3}{2}}Z^{\le N-1}u\|^2_{L^2_tL^2_x},$$

as desired. In all three cases, the δ appears as a result of the bound on the first term in (2.13). Arguing similarly to (2.17), using the bound on the last two terms of (2.13), results in

$$\int_0^\infty\int \langle r\rangle^{\tilde p}\Big(|\bar\partial h|+|h^{\alpha\beta}\omega_\alpha\omega_\beta|\Big)|\partial Z^{\le N}u|\Big(|\bar\partial Z^{\le N-1}u|+\frac{|Z^{\le N-1}u|}{\langle r\rangle}\Big)\,dx\,dt$$

$$\lesssim \|\partial Z^{\le N} u\|^2_{L^\infty_t L^2_x} + \delta\Big(\|\langle r\rangle^{\frac{\tilde p-1}{2}} \not\partial Z^{\le N-1} u\|^2_{L^2_t L^2_x} + \|\langle r\rangle^{\frac{\tilde p-3}{2}} Z^{\le N-1} u\|^2_{L^2_t L^2_x}\Big).$$

Combining these bounds immediately gives (2.18). □

3. Proof of Theorem 1.1

We begin by establishing the following lemma concerning the interaction of the admissible vector fields with the null condition. Variants of this lemma are commonplace.

Lemma 3.1. *Suppose that $A^{I,\alpha\beta}_{JK}$ and $B^{I,\gamma\alpha\beta}_{JK}$ satisfy* (1.3)*. Then, on $|x| \ge 1$,*

$$|Z^{\le N}(A^{I,\alpha\beta}_{JK}\partial_\alpha u^J \partial_\beta v^K)| \lesssim |Z^{\le N}\not\partial u||Z^{\le N/2}\partial v| + |Z^{\le N/2}\not\partial u||Z^{\le N}\partial v| + |Z^{\le N}\partial u||Z^{\le N/2}\not\partial v| + |Z^{\le N/2}\partial u||Z^{\le N}\not\partial v|, \tag{3.1}$$

and

$$|Z^{\le N}(B^{I,\gamma\alpha\beta}_{JK}\partial_\gamma u^J \partial_\alpha\partial_\beta v^K)| \lesssim |Z^{\le N}\not\partial u||Z^{\le N/2+1}\partial v| + |Z^{\le N/2}\not\partial u||Z^{\le N+1}\partial v| + |Z^{\le N}\partial u|\Big(|Z^{\le N/2+1}\not\partial v| + r^{-1}|Z^{\le N/2}\partial v|\Big) + |Z^{\le N/2}\partial u|\Big(|Z^{\le N+1}\not\partial v| + r^{-1}|\partial Z^{\le N} v|\Big). \tag{3.2}$$

for any N. Moreover, for any multi-index μ with $|\mu| \le N$,

$$\begin{aligned}|Z^\mu(B^{I,\gamma\alpha\beta}_{JK}\partial_\gamma u^J \partial_\alpha\partial_\beta v^K) - B^{I,\gamma\alpha\beta}_{JK}\partial_\gamma u^J \partial_\alpha\partial_\beta Z^\mu v^K| &\lesssim |Z^{\le N}\not\partial u||Z^{\le N/2+1}\partial v| + |Z^{\le N/2}\not\partial u||Z^{\le N}\partial v| \\ &\quad + |Z^{\le N}\partial u|\Big(|Z^{\le N/2+1}\not\partial v| + r^{-1}|Z^{\le N/2}\partial v|\Big) \\ &\quad + |Z^{\le N/2}\partial u|\Big(|Z^{\le N}\not\partial v| + r^{-1}|Z^{\le N}\partial v|\Big).\end{aligned} \tag{3.3}$$

Proof. By (1.3), we have

$$A^{I,\alpha\beta}_{JK}\partial_\alpha u^J \partial_\beta v^K = A^{I,\alpha\beta}_{JK}(\partial_\alpha - \omega_\alpha\partial_r)u^J\partial_\beta v^K + A^{I,\alpha\beta}_{JK}\omega_\alpha\partial_r u^J(\partial_\beta - \omega_\beta\partial_r)v^K.$$

The result (3.1) then follows from the product rule.

We write

$$\begin{aligned}B^{I,\gamma\alpha\beta}_{JK}\partial_\gamma u^J\partial_\alpha\partial_\beta v^K &= B^{I,\gamma\alpha\beta}_{JK}(\partial_\gamma - \omega_\gamma\partial_r)u^J\partial_\alpha\partial_\beta v^K \\ &\quad + B^{I,\gamma\alpha\beta}_{JK}\omega_\gamma\partial_r u^J(\partial_\alpha - \omega_\alpha\partial_r)\partial_\beta v^K \\ &\quad + B^{I,\gamma\alpha\beta}_{JK}\omega_\gamma\omega_\alpha\partial_r u^J\partial_r(\partial_\beta - \omega_\beta\partial_r)v^K.\end{aligned}$$

We further note that

$$B^{I,\gamma\alpha\beta}_{JK}\omega_\gamma\partial_r u^J(\partial_\alpha-\omega_\alpha\partial_r)\partial_\beta v^K = B^{I,\gamma\alpha\beta}_{JK}\omega_\gamma\partial_r u^J\partial_\beta(\partial_\alpha-\omega_\alpha\partial_r)v^K + \frac{1}{r}B^{I,\gamma\alpha j}_{JK}\omega_\gamma\partial_r u^J\omega_\alpha \nabla\!\!\!/_j v^K + \frac{1}{r}B^{I,\gamma jk}_{JK}\omega_\gamma\partial_r u^J(\delta_{jk}-\omega_j\omega_k)\partial_r v^K.$$

The desired results (3.2) and (3.3) are again a result of the product rule. □

We solve (1.1) by considering an iteration where $u_0\equiv 0$ and u_k solves

$$\begin{aligned}\Box u^I_k &= A^{I,\alpha\beta}_{JK}\partial_\alpha u^J_{k-1}\partial_\beta u^K_{k-1} + B^{I,\gamma\alpha\beta}_{JK}\partial_\gamma u^J_{k-1}\partial_\alpha\partial_\beta u^K_k,\\ u_k(0,\,\cdot\,)&=f,\quad \partial_t u_k(0,\,\cdot\,)=g.\end{aligned}\tag{3.4}$$

We let $0<2p<\tilde p<2$.

Boundedness: Our first task will be to show uniform boundedness of the sequence (u_k). We set

$$\begin{aligned}M_k =&\|\langle r\rangle^{\frac{p-1}{2}}Z^{\le N}\partial\!\!\!/ u_k\|_{L^2_tL^2_x} + \|\langle r\rangle^{\frac{\tilde p-1}{2}}Z^{\le N-1}\partial\!\!\!/ u_k\|_{L^2_tL^2_x}\\ &+\|Z^{\le N}\partial u_k\|_{L^\infty_tL^2_x} + \|Z^{\le N}u_k\|_{LE^1}.\end{aligned}$$

For $k=1$, we may take $h^{I,\alpha\beta}_{JK}\equiv 0$. From (2.14) and (2.18), which are being applied to a homogeneous equation, and (1.4), there exists a constant C_0 so that

$$M_1\le C_0\varepsilon.$$

We will use induction to show that

$$M_k\le 2C_0\varepsilon,\qquad \text{for all } k\in\mathbb{N}.\tag{3.5}$$

Assuming that the bound holds at the $(k-1)$st level, we set

$$h^{I,\alpha\beta}_K = -B^{I,\gamma\alpha\beta}_{JK}\partial_\gamma u^J_{k-1}.$$

By (1.3), for any I,J,K, we have

$$h^{I,\alpha\beta}_{JK}\omega_\alpha\omega_\beta = -B^{I,\gamma\alpha\beta}_{JK}\omega_\alpha\omega_\beta\partial_\gamma u_{k-1} = -B^{I,\gamma\alpha\beta}_{JK}\omega_\alpha\omega_\beta(\partial_\gamma-\omega_\gamma\partial_r)u_{k-1}.$$

Thus, since $\tilde p<2$ and using (1.5),

$$\begin{aligned}&\|Z^{\le 3}h\|_{L^\infty_tL^2_x} + \|\langle r\rangle^{\frac{\tilde p-1}{2}}Z^{\le 2}\partial\!\!\!/ h\|_{L^2_tL^2_x} + \|\langle r\rangle^{\frac{\tilde p-1}{2}}Z^{\le 3}(\omega_\alpha\omega_\beta h^{\alpha\beta})\|_{L^2_tL^2_x}\\ &\lesssim \|Z^{\le 3}\partial u_{k-1}\|_{L^\infty_tL^2_x} + \|\langle r\rangle^{\frac{\tilde p-1}{2}}Z^{\le 3}\partial\!\!\!/ u_{k-1}\|_{L^2_tL^2_x} + \|Z^{\le 3}u_{k-1}\|_{LE^1}.\end{aligned}$$

By the inductive hypothesis, this is $\mathcal{O}(\varepsilon)$, which establishes (2.13). Thus by (2.14) and (2.18) it will suffice to establish

$$\|\langle r\rangle^{\frac{p+1}{2}}\Box_h Z^{\le N}u_k\|_{L^2_tL^2_x} + \|\langle r\rangle^{\frac{\tilde p+1}{2}}\Box_h Z^{\le N-1}u_k\|_{L^2_tL^2_x} \lesssim M^2_{k-1}+M_{k-1}M_k.\tag{3.6}$$

As noted previously the problem of multiple good derivatives is only an artifact of considering estimates for perturbations of $\Box$. With (2.13) established, Proposition 2.5 completely addresses the issue, and the subsequent argument is quite reminiscent of the simpler semilinear case.

We first notice that

$$\begin{aligned}\Box_h Z^\mu u_k =& Z^\mu (A^{I,\alpha\beta}_{JK}\partial_\alpha u^J_{k-1}\partial_\beta u^K_{k-1}) + Z^\mu (B^{I,\gamma\alpha\beta}_{JK}\partial_\gamma u^J_{k-1}\partial_\alpha\partial_\beta u^K_k) \\ &- B^{I,\gamma\alpha\beta}_{JK}\partial_\gamma u^J_{k-1}\partial_\alpha\partial_\beta Z^\mu u^K_k.\end{aligned}$$

By (3.1) and (3.3), it follows that

$$\begin{aligned}(3.7)\quad & \|\langle r\rangle^{\frac{p+1}{2}}\Box_h Z^{\le N} u_k\|_{L^2_tL^2_x} \lesssim \|\langle r\rangle^{\frac{p+1}{2}} |Z^{\le N}\bar\partial u_{k-1}||Z^{\le \frac{N}{2}}\partial u_{k-1}|\|_{L^2_tL^2_x} \\ &+ \|\langle r\rangle^{\frac{p+1}{2}} |Z^{\le \frac{N}{2}}\bar\partial u_{k-1}||Z^{\le N}\partial u_{k-1}|\|_{L^2_tL^2_x} + \|\langle r\rangle^{\frac{p+1}{2}} |Z^{\le N}\bar\partial u_{k-1}||Z^{\le \frac{N}{2}+1}\partial u_{k}|\|_{L^2_tL^2_x} \\ &+ \|\langle r\rangle^{\frac{p+1}{2}} |Z^{\le \frac{N}{2}}\bar\partial u_{k-1}||Z^{\le N}\partial u_{k}|\|_{L^2_tL^2_x} + \|\langle r\rangle^{\frac{p+1}{2}} |Z^{\le N}\partial u_{k-1}||Z^{\le \frac{N}{2}+1}\bar\partial u_{k}|\|_{L^2_tL^2_x} \\ &+ \|\langle r\rangle^{\frac{p+1}{2}} |Z^{\le \frac{N}{2}}\partial u_{k-1}||Z^{\le N}\bar\partial u_{k}|\|_{L^2_tL^2_x} + \|\langle r\rangle^{\frac{p-1}{2}} |Z^{\le \frac{N}{2}}\partial u_{k-1}||Z^{\le N}\partial u_{k-1}|\|_{L^2_tL^2_x} \\ &+ \|\langle r\rangle^{\frac{p-1}{2}} |Z^{\le N}\partial u_{k-1}||Z^{\le \frac{N}{2}+1}\partial u_{k}|\|_{L^2_tL^2_x} + \|\langle r\rangle^{\frac{p-1}{2}} |Z^{\le \frac{N}{2}}\partial u_{k-1}||Z^{\le N}\partial u_{k}|\|_{L^2_tL^2_x}.\end{aligned}$$

We notice that the last three terms provide the appropriate bounds when $|x| \le 1$. For each of the first six terms in the right, we apply (2.12) to the term with fewer vector fields and measure the good derivative factor in a weighted $L^2_tL^2_x$-space and the other factor in an energy space $L^\infty_tL^2_x$. Provided $\frac{N}{2}+3 \le N$, by (2.12), we have

$$\begin{aligned}(3.8)\qquad & \|\langle r\rangle^{\frac{p+1}{2}} Z^{\le N}w\, Z^{\le \frac{N}{2}+1}v\|_{L^2_tL^2_x} + \|\langle r\rangle^{\frac{p+1}{2}} Z^{\le \frac{N}{2}+1}w\, Z^{\le N}v\|_{L^2_tL^2_x} \\ &\lesssim \|\langle r\rangle^{\frac{p-1}{2}} Z^{\le N}w\|_{L^2_tL^2_x}\|Z^{\le N}v\|_{L^\infty_tL^2_x}\end{aligned}$$

and, since $p < 2$,

$$(3.9)\qquad \|\langle r\rangle^{\frac{p-1}{2}} Z^{\le N}w\, Z^{\le \frac{N}{2}+1}v\|_{L^2_tL^2_x} \lesssim \|Z^{\le N}w\|_{L^\infty_tL^2_x}\|Z^{\le N}v\|_{LE}.$$

Indeed, (3.9) follows as the square of the left side is controlled by

$$\sum_{j\ge 0} 2^{(p-1)j}\|Z^{\le N}w\, Z^{\le \frac{N}{2}+1}v\|^2_{L^2_tL^2_x(\mathbb{R}_+\times\{2^{j-1}\le\langle x\rangle\le 2^j\})},$$

which after an application of (2.12) is

$$\lesssim \Big(\sum_{j\ge 0} 2^{p-2}\Big)\|Z^{\le N}w\|^2_{L^\infty_tL^2_x}\sup_{j\ge 0} 2^{-j}\|Z^{\le \frac{N}{2}+3}v\|^2_{L^2_tL^2_x(\mathbb{R}_+\times\{2^{j-2}\le\langle x\rangle\le 2^{j+1}\})},$$

from which (3.9) follows readily. Using (3.8) and (3.9) repeatedly, it follows that the right side of (3.7) is

$$\lesssim M_{k-1}^2 + M_{k-1}M_k.$$

This provides the bound for the first term in the left side of (3.6). The bound for the second term is nearly identical where all of the p are replaced by $\tilde{p}$ and the N by $N-1$.

Convergence: We now establish that the sequence (u_k) is Cauchy. It, thus, converges, and its limit is the desired solution.

To this end, we set

$$A_k = \|\langle r\rangle^{\frac{p-1}{2}} Z^{\le N-1} \not\partial(u_k - u_{k-1})\|_{L^2_t L^2_x} + \|Z^{\le N-1}\partial(u_k - u_{k-1})\|_{L^\infty_t L^2_x} + \|Z^{\le N-1}(u_k - u_{k-1})\|_{LE^1}. \tag{3.10}$$

We will prove that

$$A_k \le \frac{1}{2}A_{k-1} \quad \text{for all } k. \tag{3.11}$$

We begin by noting

$$\begin{aligned}\Box(u_k^I - u_{k-1}^I) = A^{I,\alpha\beta}_{JK}\partial_\alpha(u^J_{k-1} - u^J_{k-2})\partial_\beta u^K_{k-1} \\ + A^{I,\alpha\beta}_{JK}\partial_\alpha u^J_{k-2}\partial_\beta(u^K_{k-1} - u^K_{k-2}) + B^{I,\gamma\alpha\beta}_{JK}\partial_\gamma u^J_{k-1}\partial_\alpha\partial_\beta(u^K_k - u^K_{k-1}) \\ + B^{I,\gamma\alpha\beta}_{JK}\partial_\gamma(u^J_{k-1} - u^J_{k-2})\partial_\alpha\partial_\beta u^K_{k-1}.\end{aligned}$$

With $h^{I,\alpha\beta}_K = -B^{I,\gamma\alpha\beta}_{JK}\partial_\gamma u^J_{k-1}$, as above, (3.5) implies (2.13). We may, thus, apply (2.14). Since $u_k - u_{k-1}$ has vanishing Cauchy data, it suffices to bound

$$\|\langle r\rangle^{\frac{p+1}{2}}\Box_h Z^{\le N-1}(u_k - u_{k-1})\|_{L^2_t L^2_x} \lesssim (M_{k-1} + M_{k-2})A_{k-1} + M_{k-1}A_k \tag{3.12}$$

as we may then apply (3.5) and absorb A_k to the other side, which will yield (3.11) as long as ε is sufficiently small.

Using (3.1), (3.3), and (3.2), we have

$$\begin{aligned}&\|\langle r\rangle^{\frac{p+1}{2}}\Box_h Z^{\le N-1}(u_k - u_{k-1})\|_{L^2_t L^2_x} \\ &\lesssim \Big\|\langle r\rangle^{\frac{p+1}{2}}|Z^{\le N-1}\not\partial(u_{k-1} - u_{k-2})|\Big(|Z^{\le\frac{N+1}{2}}\partial u_{k-1}| + |Z^{\le\frac{N-1}{2}}\partial u_{k-2}|\Big)\Big\|_{L^2_t L^2_x} \\ &\quad + \Big\|\langle r\rangle^{\frac{p+1}{2}}|Z^{\le\frac{N-1}{2}}\not\partial(u_{k-1} - u_{k-2})|\Big(|Z^{\le N}\partial u_{k-1}| + |Z^{\le N-1}\partial u_{k-2}|\Big)\Big\|_{L^2_t L^2_x} \\ &\quad + \Big\|\langle r\rangle^{\frac{p+1}{2}}|Z^{\le N-1}\partial(u_{k-1} - u_{k-2})|\Big(|Z^{\le\frac{N+1}{2}}\not\partial u_{k-1}| + |Z^{\le\frac{N-1}{2}}\not\partial u_{k-2}|\Big)\Big\|_{L^2_t L^2_x} \\ &\quad + \Big\|\langle r\rangle^{\frac{p+1}{2}}|Z^{\le\frac{N-1}{2}}\partial(u_{k-1} - u_{k-2})|\Big(|Z^{\le N}\not\partial u_{k-1}| + |Z^{\le N-1}\not\partial u_{k-2}|\Big)\Big\|_{L^2_t L^2_x}\end{aligned} \tag{3.13}$$

$$
\begin{aligned}
&+\Big\|\langle r\rangle^{\frac{p+1}{2}}|Z^{\le\frac{N+1}{2}}\partial(u_k-u_{k-1})||Z^{\le N-1}\bar\partial u_{k-1}|\Big\|_{L^2_tL^2_x}\\
&+\Big\|\langle r\rangle^{\frac{p+1}{2}}|Z^{\le N-1}\partial(u_k-u_{k-1})||Z^{\le\frac{N-1}{2}}\bar\partial u_{k-1}|\Big\|_{L^2_tL^2_x}\\
&+\Big\|\langle r\rangle^{\frac{p+1}{2}}|Z^{\le\frac{N+1}{2}}\bar\partial(u_k-u_{k-1})||Z^{\le N-1}\partial u_{k-1}|\Big\|_{L^2_tL^2_x}\\
&+\Big\|\langle r\rangle^{\frac{p+1}{2}}|Z^{\le N-1}\bar\partial(u_k-u_{k-1})||Z^{\le\frac{N-1}{2}}\partial u_{k-1}|\Big\|_{L^2_tL^2_x}\\
&+\Big\|\langle r\rangle^{\frac{p-1}{2}}|Z^{\le N-1}\partial(u_{k-1}-u_{k-2})|\Big(|Z^{\le\frac{N+1}{2}}\partial u_{k-1}|+|Z^{\le\frac{N-1}{2}}\partial u_{k-2}|\Big)\Big\|_{L^2_tL^2_x}\\
&+\Big\|\langle r\rangle^{\frac{p-1}{2}}|Z^{\le\frac{N-1}{2}}\partial(u_{k-1}-u_{k-2})|\Big(|Z^{\le N}\partial u_{k-1}|+|Z^{\le N-1}\partial u_{k-2}|\Big)\Big\|_{L^2_tL^2_x}\\
&+\Big\|\langle r\rangle^{\frac{p-1}{2}}|Z^{\le N-1}\partial(u_k-u_{k-1})||Z^{\le\frac{N-1}{2}}\partial u_{k-1}|\Big\|_{L^2_tL^2_x}\\
&+\Big\|\langle r\rangle^{\frac{p-1}{2}}|Z^{\le\frac{N+1}{2}}\partial(u_k-u_{k-1})||Z^{\le N-1}\partial u_{k-1}|\Big\|_{L^2_tL^2_x}.
\end{aligned}
$$

We proceed with an argument that is akin to that used in the proof of (3.5). For each term, we apply (2.12) to the lower order factor. We then measure the "good" derivative factor in a weighted $L^2_tL^2_x$ space, while the other factor is placed into an energy space. This approach, which is based in (3.8) and (3.9), shows that the first four terms in the right side of (3.13) are bounded by

$$
\begin{aligned}
&\|\langle r\rangle^{\frac{p-1}{2}}Z^{\le N-1}\bar\partial(u_{k-1}-u_{k-2})\|_{L^2_tL^2_x}\\
&\quad\times\Big(\|Z^{\le\frac{N+5}{2}}\partial u_{k-1}\|_{L^\infty_tL^2_x}+\|Z^{\le\frac{N+3}{2}}\partial u_{k-2}\|_{L^\infty_tL^2_x}\Big)\\
&\quad+\|\langle r\rangle^{\frac{p-1}{2}}|Z^{\le\frac{N+3}{2}}\bar\partial(u_{k-1}-u_{k-2})|\|_{L^2_tL^2_x}\\
&\quad\times\Big(\|Z^{\le N}\partial u_{k-1}\|_{L^\infty_tL^2_x}+|Z^{\le N-1}\partial u_{k-2}\|_{L^\infty_tL^2_x}\Big)\\
&\quad+\|Z^{\le N-1}\partial(u_{k-1}-u_{k-2})\|_{L^\infty_tL^2_x}\\
&\quad\times\Big(\|\langle r\rangle^{\frac{p-1}{2}}Z^{\le\frac{N+5}{2}}\bar\partial u_{k-1}\|_{L^2_tL^2_x}+\|\langle r\rangle^{\frac{p-1}{2}}Z^{\le\frac{N+3}{2}}\bar\partial u_{k-2}\|_{L^2_tL^2_x}\Big)\\
&\quad+\|Z^{\le\frac{N+3}{2}}\partial(u_{k-1}-u_{k-2})\|_{L^\infty_tL^2_x}\\
&\quad\times\Big(\|\langle r\rangle^{\frac{p-1}{2}}Z^{\le N}\bar\partial u_{k-1}\|_{L^2_tL^2_x}+\|\langle r\rangle^{\frac{p-1}{2}}Z^{\le N-1}\bar\partial u_{k-2}\|_{L^2_tL^2_x}\Big),
\end{aligned}
$$

which, provided that $\frac{N+5}{2}\le N$, is

$$\lesssim A_{k-1}\Big(M_{k-1}+M_{k-2}\Big).$$

Similarly, the fifth through eighth terms in the right side of (3.13) can be controlled by

$$\begin{aligned}\|Z^{\le\frac{N+5}{2}}\partial(u_k-u_{k-1})\|_{L^\infty_t L^2_x}\|\langle r\rangle^{\frac{p-1}{2}}Z^{\le N-1}\bar\partial u_{k-1}\|_{L^2_tL^2_x}\\ +\|Z^{\le N-1}\partial(u_k-u_{k-1})\|_{L^\infty_tL^2_x}\|\langle r\rangle^{\frac{p-1}{2}}Z^{\le\frac{N+3}{2}}u_{k-1}\|_{L^2_tL^2_x}\\ +\|\langle r\rangle^{\frac{p-1}{2}}|Z^{\le\frac{N+5}{2}}\bar\partial(u_k-u_{k-1})\|_{L^2_tL^2_x}\|Z^{\le N-1}\partial u_{k-1}|\|_{L^\infty_tL^2_x}\\ +\|\langle r\rangle^{\frac{p-1}{2}}|Z^{\le N-1}\bar\partial(u_k-u_{k-1})\|_{L^2_tL^2_x}\|Z^{\le\frac{N-1}{2}}\partial u_{k-1}\|_{L^\infty_tL^2_x},\end{aligned}$$

which in turn is $\lesssim A_k\cdot M_{k-1}$ provided that $\frac{N+5}{2}\le N-1$. Relying on the fact that $p<2$, the remaining terms in (3.13) (namely the last four in the right side) are

$$\begin{aligned}\lesssim\|Z^{\le\frac{N+3}{2}}\partial(u_{k-1}-u_{k-2})\|_{L^\infty_tL^2_x}\Big(\|Z^{\le N}u_{k-1}\|_{LE^1}+\|Z^{\le N-1}u_{k-2}\|_{LE^1}\Big)\\ +\|Z^{\le N-1}\partial(u_{k-1}-u_{k-2})\|_{L^\infty_tL^2_x}\Big(\|Z^{\le\frac{N+5}{2}}u_{k-1}\|_{LE^1}+\|Z^{\le\frac{N+3}{2}}u_{k-2}\|_{LE^1}\Big)\\ +\|Z^{\le N-1}\partial(u_k-u_{k-1})\|_{L^\infty_tL^2_x}\|Z^{\le\frac{N+3}{2}}u_{k-1}\|_{LE^1}\\ +\|Z^{\le\frac{N+5}{2}}\partial(u_k-u_{k-1})\|_{L^\infty_tL^2_x}\|Z^{\le N-1}u_{k-1}\|_{LE^1}\end{aligned}$$

As this is

$$\lesssim\Big(M_{k-2}+M_{k-1}\Big)A_{k-1}+M_{k-1}\cdot A_k,$$

we have completed the proof of (3.12), which also completes the proof of Theorem 1.1.

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Received December 16, 2021
Revised version received June 15, 2022

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