Local Energy Estimate for Damped Waves

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Contents

1 Introduction

In this paper, we will prove a local energy estimate which is applicable for the case of a damped wave equation.

We define the following: the wave operator $\Box \equiv \partial_t^2 - \Delta$, the angular derivative $\not\nabla \equiv \nabla - \frac{x}{r}\partial_r$, the "Japanese bracket" $\langle x\rangle \equiv \sqrt{1+|x|^2}$, and dyadic regions $X_U \equiv \{(t,x) : U \leq \langle t-r\rangle \leq 2U\}$.

Theorem 1.1. *For a function $u \in C^2(\mathbb{R}_+ \times \mathbb{R}^3)$, where for all $T > 0$ there exists some $R > 0$ such that $|u(t,x)| = 0$ for all $t \in [0,T)$ and $|x| > R$, and for positive constant a and constant p such that $0 < p < 1$,*

$$\begin{aligned}
&\left\|\langle r\rangle^{\frac{p-1}{2}} r^{-1}(\partial_t+\partial_r)(ru)\right\|_{L^2L^2}^2 + \left\|\langle r\rangle^{\frac{p}{2}} \not\nabla u\right\|_{L^\infty L^2}^2 + \left\|\langle r\rangle^{\frac{p}{2}} r^{-\frac{1}{2}} \not\nabla u\right\|_{L^2L^2}^2 \\
&\quad + a\left\|\langle r\rangle^{\frac{p}{2}} r^{-1}(\partial_t+\partial_r)(ru)\right\|_{L^2L^2}^2 + \left\|\langle r\rangle^{\frac{p}{2}} r^{-1}\left((\partial_t+\partial_r)(ru)+\frac{a}{2}(ru)\right)\right\|_{L^\infty L^2}^2 \\
&\quad + a^2\left\|\langle r\rangle^{\frac{p}{2}} u\right\|_{L^\infty L^2}^2 + \sup_U \left\|\langle r\rangle^{\frac{p}{2}} r^{-1}\langle t-r\rangle^{-\frac{1}{2}}\left((\partial_t+\partial_r)(ru)+\frac{a}{2}(ru)\right)\right\|_{L^2L^2(X_U)}^2 \\
&\quad + a\left\|\langle r\rangle^{\frac{p-1}{2}} u\right\|_{L^\infty L^2}^2 + a\left\|\langle r\rangle^{\frac{p-2}{2}} u\right\|_{L^2L^2}^2 + a\left\|\langle r\rangle^{\frac{p}{2}} \not\nabla u\right\|_{L^2L^2}^2 \\
&\lesssim \left\|\langle r\rangle^{\frac{p}{2}} \not\nabla u(0,\cdot)\right\|_{L^2}^2 + \left\|\langle r\rangle^{\frac{p}{2}} r^{-1}\left((\partial_t+\partial_r)(ru(0,\cdot))+\frac{a}{2}(ru(0,\cdot))\right)\right\|_{L^2}^2 \\
&\quad + a^2\left\|\langle r\rangle^{\frac{p}{2}} u(0,\cdot)\right\|_{L^2}^2 + a\left\|\langle r\rangle^{\frac{p-1}{2}} u(0,\cdot)\right\|_{L^2}^2 + \left(\sum_U U^{\frac{1}{2}}\left\|\langle r\rangle^{\frac{p}{2}}(\Box + a\partial_t)u\right\|_{L^2L^2(X_U)}\right)^2.
\end{aligned}$$

This estimate is based on an estimate which was previously proven in [5, 3] and used methods from [2] and [1], and was used in [5] to show global existence for a system of weakly null semilinear equations.

To prove that a solution exists one must show that a sequence converging to it is bounded and Cauchy in some norm. Therefore such proofs require a certain "smallness" of the system to be demonstrated, i.e. that the system decays over time and space. Estimates, such as the one in [5, 3], show certain quantities are bounded, and are thus used to demonstrate "smallness" of the solutions in existence proofs.

The new estimate in this paper expands on the previous one by taking into account the effects of damping. Damping forces are a source of decay, decreasing the amplitude of waves over time. Therefore we expect it to be easier to demonstrate sufficient "smallness" of damped wave systems, and anticipate the damped wave estimate to be an improvement from the original. In fact we do recover an improved bound on the damped wave. This has applications to existence proofs for damped wave systems.

In our calculations, we will prove the estimate with a constant factor on the damping force. We will examine the case of the undamped system, where the constant is equal to zero, to recover the original estimate in [5, 3]. We will then compare it to the case of the damped system, where the constant is equal to one, and discuss the improvements to the lower bound that appear.

2 Background

2.1 The Wave Operator

The wave operator, also called the d'Alembertian, is defined by

$$\Box = \partial_t^2 - \triangle$$

where $\triangle$ is the Laplacian. The d'Alembertian operates on a function $u(t, x)$ where t is time and restricted to $\mathbb{R}_+$, and x is a real vector in n-dimensional space. For this paper we will be dealing only with $n = 3$. The function u is assumed to be twice continuously differentiable, or C^2.

The homogeneous wave equation $\Box u = 0$ represents the case of a free wave in a flat geometry. Terms on the right-hand side correspond to forces applied to the system. In this paper, we examine the case where the force applied is a damping force.

An example of a damping force is drag, which slows an object down as it moves through a medium. Waves in three dimensions disperse and reduce in amplitude over time; for damped waves, the amplitude is further reduced by the damping force. Damping is usually modelled as a force that is proportional to and opposite velocity, so it is represented by a linear term which we add to the wave operator:

$$\Box + a\partial_t.$$

When the wave operator acts on a function u, it can also be helpful to define it in terms of two operators: the radial derivative ∂_r defined by $\frac{x}{r} \cdot \nabla$, and angular derivative $\not\nabla$ defined via

$$\nabla = \frac{x}{r}\partial_r + \not\nabla$$

which is an orthogonal decomposition of the gradient into a radial component and an angular component. This orthogonality is demonstrated by

$$\frac{x}{r} \cdot \not\nabla w = \sum_{i=1}^{3} \frac{x_i}{r} \cdot \not\nabla_i w = \sum_{i=1}^{3} \frac{x_i}{r} \cdot (\partial_{x_i} w - \frac{x_i}{r}\partial_r w) = \partial_r w - \partial_r w = 0. \tag{1}$$

In the following calculation, we will rewrite $\Box u$ in terms of the radial and angular derivatives.

Lemma 2.1.
$$\Box u = r^{-1}\left(\partial_t^2(ru) - \partial_r^2(ru) - \not\nabla\cdot\not\nabla(ru)\right)$$

Proof. We begin with the definition of the wave operator and expand the gradient into radial and angular components:

$$\begin{aligned}\Box u &= \partial_t^2 u - (\frac{x}{r}\partial_r + \not\nabla)\cdot(\frac{x}{r}\partial_r + \not\nabla)u \\ &= \partial_t^2 u - (\frac{x}{r}\partial_r)\cdot(\frac{x}{r}\partial_r)u - (\frac{x}{r}\partial_r)\cdot\not\nabla u - \not\nabla\cdot(\frac{x}{r}\partial_r)u - \not\nabla\cdot\not\nabla u.\end{aligned}$$

The vector $\frac{x}{r}$ is a unit vector and thus independent of radius, so it will commute with ∂_r. Using the fact that $x\cdot x = r^2$, the second term can be simplified.

$$= \partial_t^2 u - \partial_r^2 u - (\frac{x}{r}\partial_r)\cdot\not\nabla u - \not\nabla\cdot(\frac{x}{r}\partial_r)u - \not\nabla\cdot\not\nabla u.$$

Next we calculate the third and fourth terms. By (1), the third term is

$$-\frac{x}{r}\partial_r\cdot\not\nabla u = -\partial_r\left(\frac{x}{r}\cdot\not\nabla u\right) = 0 \tag{2}$$

and the fourth term is

$$\begin{aligned}-\not\nabla\cdot(\frac{x}{r}\partial_r u) &= -(\not\nabla\cdot\frac{x}{r})\partial_r u - \frac{x}{r}\cdot\not\nabla(\partial_r u) \\ &= -(\not\nabla\cdot\frac{x}{r})\partial_r u \\ &= -\sum_{i=1}^{3}(\not\nabla_i\frac{x_i}{r})\partial_r u \\ &= -\sum_{i=1}^{3}(\partial_{x_i}\frac{x_i}{r})\partial_r u + \sum_{i=1}^{3}\frac{x_i}{r}\partial_r(\frac{x_i}{r})\partial_r u \\ &= -\sum_{i=1}^{3}\frac{1}{r}\partial_r u - \sum_{i=1}^{3}x_i(\frac{-1}{r^2})\frac{x_i}{r}\partial_r u \\ &= -\frac{3}{r}\partial_r u + \frac{1}{r}\partial_r u \\ &= -\frac{2}{r}u.\end{aligned}$$

Combining these terms together gives

$$\Box u = \partial_t^2 u - \partial_r^2 u - \frac{2}{r}\partial_r u - \not\nabla\cdot\not\nabla u.$$

Then we write $u = r^{-1}ru$ to yield

$$\begin{aligned}\Box u &= (\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r - \not\nabla\cdot\not\nabla)\left(r^{-1}\,ru\right) \\ &= \partial_t^2(r^{-1}\,ru) - \partial_r^2(r^{-1}\,ru) - \frac{2}{r}\partial_r(r^{-1}\,ru) - \not\nabla\cdot\not\nabla(r^{-1}\,ru).\end{aligned} \tag{3}$$

The partial derivative with respect to time ∂_t is independent of spatial variables, and r^{-1} can be pulled out of the first term

$$\partial_t^2(r^{-1}\, ru) = r^{-1}\, \partial_t^2(ru).$$

The angular derivative is also independent of radius, which is demonstrated below:

$$\begin{aligned}\not\nabla r &= \left(\nabla - \frac{x}{r}\partial_r\right) r = \nabla\left(x_1^2 + x_2^2 + x_3^2\right)^{\frac{1}{2}} - \frac{x}{r} \\ &= 2x\frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)^{-\frac{1}{2}} - \frac{x}{r} = \frac{x}{r} - \frac{x}{r} = 0.\end{aligned}$$

Therefore r^{-1}, as a function of only r, can be pulled out of the angular derivative term of (3) as well:

$$-\not\nabla \cdot \not\nabla(r^{-1}\, ru) = -r^{-1}\not\nabla \cdot \not\nabla(ru).$$

Now we expand the second term of (3),

$$\begin{aligned}-\partial_r^2(r^{-1}\, ru) &= -\partial_r\left(\partial_r(r^{-1}\, ru)\right) \\ &= -\partial_r\left(-r^{-2}(ru) + r^{-1}\,\partial_r(ru)\right) \\ &= -2r^{-3}(ru) + r^{-2}\partial_r(ru) + r^{-2}\partial_r(ru) - r^{-1}\,\partial_r^2(ru) \\ &= -\frac{2}{r^3}(ru) + \frac{2}{r^2}\partial_r(ru) - r^{-1}\,\partial_r^2(ru)\end{aligned}$$

as well as the third term,

$$\begin{aligned}-\frac{2}{r}\partial_r(r^{-1}\, ru) &= -\frac{2}{r}\left(-r^{-2}(ru) + r^{-1}\partial_r(ru)\right) \\ &= \frac{2}{r^3}(ru) - \frac{2}{r^2}\partial_r(ru).\end{aligned}$$

Substituting all pieces back into (3), we notice some like terms which can be cancelled, completing the proof:

$$\Box u = r^{-1}\left(\partial_t^2(ru) - \partial_r^2(ru) - \not\nabla \cdot \not\nabla(ru)\right).$$

■

We will also prove a useful result for the commutator of the angular derivative $\not\nabla$ with the radial derivative ∂_r.

Lemma 2.2.

$$\left[\not\nabla, \partial_r\right] = [\nabla, \partial_r] = \frac{1}{r}\not\nabla$$

Proof. We begin by computing

$$\begin{aligned}\left[\not\nabla, \partial_r\right] &= \not\nabla\partial_r - \partial_r\not\nabla \\ &= (\nabla - \frac{x}{r}\partial_r)\partial_r - \partial_r(\nabla - \frac{x}{r}\partial_r) \\ &= \nabla\partial_r - \frac{x}{r}\partial_r^2 - \partial_r\nabla + \partial_r(\frac{x}{r}\partial_r).\end{aligned}$$

The vector $\frac{x}{r}$ is always a unit vector, and is thus independent of radius. Therefore $\frac{x}{r}$ can be pulled out of the radial derivative in the last term:

$$
\begin{aligned}
&= \nabla \partial_r - \frac{x}{r}\partial_r^2 - \partial_r \nabla + \frac{x}{r}\partial_r^2 \\
&= \nabla \partial_r - \partial_r \nabla = [\nabla, \partial_r]\,.
\end{aligned}
$$

To calculate the commutator of the gradient and radial derivative, we will examine one component of the gradient

$$[\nabla_j, \partial_r] = \left[\partial_{x_j}, \partial_r\right] = \partial_{x_j}\partial_r - \partial_r \partial_{x_j}$$

and let it operate on a function u

$$[\nabla_j, \partial_r]\, u = \partial_{x_j}\left(\partial_r\, u\right) - \partial_r \left(\partial_{x_j}\, u\right).$$

We will rewrite the radial derivative ∂_r as $\sum_{n=1}^{3} \frac{x_n}{r}\partial_{x_n}$:

$$
\begin{aligned}
&= \partial_{x_j}\left(\sum_{n=1}^{3} \frac{x_n}{r}\partial_{x_n}\, u\right) - \sum_{m=1}^{3} \frac{x_m}{r}\partial_{x_m}\partial_{x_j}\, u \\
&= \left(\sum_{n=1}^{3} \partial_{x_j}\frac{x_n}{r}\right)\partial_{x_n}\, u + \sum_{n=1}^{3} \frac{x_n}{r}\partial_{x_j}\partial_{x_n}\, u - \sum_{m=1}^{3} \frac{x_m}{r}\partial_{x_m}\partial_{x_j}\, u \\
&= \left(\sum_{n=1}^{3} \partial_{x_j}\frac{x_n}{r}\right)\partial_{x_n}\, u.
\end{aligned}
$$

The partial derivative will be different depending on whether or not $j = n$. We divide the sum into two parts

$$
\begin{aligned}
&= \partial_{x_j}\left(\frac{x_j}{r}\right)\partial_{x_j} u + \sum_{n\neq j} \partial_{x_j}\left(\frac{x_n}{r}\right)\partial_{x_n}\, u \\
&= \frac{1}{r}\partial_{x_j} + x_j \frac{-1}{r^2}\frac{x_j}{r}\partial_{x_j} u + \sum_{n\neq j} x_n \left(\frac{-1}{r^2}\right)\frac{x_j}{r}\partial_{x_n} u,
\end{aligned}
$$

and then can recombine the second and third term into one summation

$$
\begin{aligned}
&= \frac{1}{r}\partial_{x_j} - \frac{x_j}{r^2}\sum_{n=1}^{3} \frac{x_n}{r}\partial_{x_n} u \\
&= \frac{1}{r}\partial_{x_j} - \frac{x_j}{r^2}\partial_r u.
\end{aligned}
$$

Using the definition of the angular derivative, this becomes

$$= \frac{1}{r}\not{\nabla}_j u$$

which completes the proof.

$\blacksquare$

2.2 Local Energy Estimates

The estimate in [5, 3] and its extension in this paper are both local energy type estimates. The energy of a wave is defined as

$$E[u](t) = \frac{1}{2}\int |u'(t,x)|^2 dx = \frac{1}{2}||u'(t,x)||_{L^2}^2$$

where u' is the vector $\langle \partial_t u, \nabla u\rangle$.

A basic example of an energy estimate is conservation of energy. For the homogeneous wave equation, with no external forces acting on the system, no work is being done. Therefore we expect energy conservation, meaning that the energy at any time T is equal to the energy at time 0.

Theorem 2.3. *For the three-dimensional homogeneous wave equation $\Box u = 0$, assuming that for all $T > 0$ there exists some $R > 0$ such that $|u(t,x)| = 0$ for all $t \in [0,T)$ and $|x| > R$, energy is conserved. I.e., for any time $T > 0$,*

$$E[u](T) = E[u](0).$$

Proof. We start with the integral

$$0 = \int_0^T \int_{\mathbb{R}^3} \Box u \, \partial_t u \, dxdt.$$

Intuitively, the reason for including the $\partial_t u$ multiplier is that conservation of energy corresponds to time translation symmetry. In various estimates we will see multipliers which take advantage of specific symmetries.

We expand the above equation to obtain

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}^3} \Box u \, \partial_t u \, dxdt \\ &= \int_0^T \int_{\mathbb{R}^3} \left[\partial_t^2 u - \sum_{i=1}^3 \partial_{x_i}^2 u\right] \partial_t u \, dxdt \\ &= \int_0^T \int_{\mathbb{R}^3} \partial_t^2 u \, \partial_t u \, dxdt - \sum_{i=1}^3 \int_0^T \int_{\mathbb{R}^3} \partial_{x_i}^2 u \, \partial_t u \, dxdt. \end{aligned}$$

On the second term we will integrate by parts on ∂_{x_i}, noting that ∂_{x_i} and ∂_t commute:

$$= \int_0^T \int_{\mathbb{R}^3} \partial_t^2 u \, \partial_t u \, dxdt + \sum_{i=1}^3 \int_0^T \int_{\mathbb{R}^3} \partial_{x_i} u \, \partial_t \partial_{x_i} u \, dxdt.$$

These can be re-written using the chain rule relation $v \, \mathrm{d}v = \frac{1}{2}\,\mathrm{d}(v)^2$ to obtain

$$= \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} \left[\partial_t(\partial_t u)^2 + \sum_{i=1}^3 \partial_t(\partial_{x_i} u)^2\right] dxdt,$$

which, by the Fundamental Theorem of Calculus, is

$$= \frac{1}{2}\int_{\mathbb{R}^3} |u'(t,x)|^2 dx\Big|_0^T = E[u](T) - E[u](0)$$

$$\implies E[u](T) = E[u](0).$$

■

We modify this calculation to demonstrate loss of energy in the damped case.

Theorem 2.4. *For the three-dimensional wave equation* $(\Box + a\partial_t)u = 0$ *where* $a \geq 0$*, assuming that for all* $T > 0$ *there exists some* $R > 0$ *such that* $|u(t,x)| = 0$ *for all* $t \in [0,T)$ *and* $|x| > R$*, then for any time* $T > 0$*,*

$$E[u](T) \lesssim E[u](0).$$

Proof. The calculation is similar to that of the free wave, but an additional term is introduced,

$$\begin{aligned}
0 &= \int_0^T \int_{\mathbb{R}^3} (\Box + a\partial_t)u\,\partial_t u\,dxdt \\
&= E[u](T) - E[u](0) + a\int_0^T \int_{\mathbb{R}^3} (\partial_t u)^2\,dxdt \\
\implies E[u](T) &+ a\int_0^T \int_{\mathbb{R}^3} (\partial_t u)^2\,dxdt = E[u](0).
\end{aligned}$$

The integral is a non-negative quantity, and therefore we show that

$$\implies E[u](T) \lesssim E[u](0).$$

■

This example explicitly demonstrates how damping introduces additional decay to the solution.

2.3 $f(r)$-Weighted Estimate

We will now examine an estimate which has some function $f(r)$ as a weight, and look for what qualities the function should possess to return a helpful estimate. We will see later in this process that the case of $f(r) = 1$ returns a well-known result.

Theorem 2.5. *For the three-dimensional homogeneous wave equation* $\Box u = 0$*, assuming that for all* $T > 0$ *there exists some* $R > 0$ *such that* $|u(t,x)| = 0$ *for all* $t \in [0,T)$ *and* $|x| > R$*, then*

$$\left\| r^{-\frac{1}{2}} \not\nabla u \right\|_{L^2L^2}^2 \lesssim E[u](0)$$

and, for spacetime regions $A_R \equiv \{(t,x) : R \leq \langle x \rangle \leq 2R\}$*,*

$$\sup_R \left\| \frac{u'}{\langle x \rangle^{\frac{1}{2}}} \right\|_{L^2L^2(A_R)}^2 + \sup_R \left\| \frac{u}{\langle x \rangle^{\frac{3}{2}}} \right\|_{L^2L^2(A_R)}^2 \lesssim E[u](0).$$

The first result is known as the Morawetz estimate [6], while the second result is found in [4] and [7].

Proof. We begin with the following integral,

$$\int_0^T \int_{\mathbb{R}^3} f(r)\,\Box u\,\left(\partial_r + \frac{1}{r}\right) u\,dxdt.$$

This calculation will result in many terms, and our goal will be to put them in forms which are of a known sign. This will allow us to more easily manipulate the final inequality.

Expanding the $\Box$ operator yields

$$= \int_0^T \int_{\mathbb{R}^3} (\partial_t^2 u - \sum_{i=1}^3 \partial_{x_i}^2 u)\, f(r) \left[\partial_r u + \frac{1}{r} u\right] dxdt. \tag{4}$$

We will examine the first term, which evaluates to

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} \partial_t^2 u \left[f(r)\partial_r u + \frac{f(r)}{r} u\right] dxdt \\
&\quad = \int_{\mathbb{R}^3} \partial_t u \left[f(r)\partial_r u + \frac{f(r)}{r} u\right] dx\Big|_0^T - \int_0^T \int_{\mathbb{R}^3} \partial_t u \left[f(r)\partial_r \partial_t u + \frac{f(r)}{r} \partial_t u\right] dxdt \\
&\quad = \int_{\mathbb{R}^3} f(r)\partial_t u \partial_r u\,dx\Big|_0^T + \int_{\mathbb{R}^3} \frac{f(r)}{r} u \partial_t u\,dx\Big|_0^T \\
&\qquad - \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} f(r)\partial_r (\partial_t u)^2\,dxdt - \int_0^T \int_{\mathbb{R}^3} \frac{f(r)}{r} (\partial_t u)^2\,dxdt.
\end{aligned}$$

On the third term, we convert to spherical coordinates so that we can perform integration by parts on ∂_r:

$$\begin{aligned}
&-\frac{1}{2} \int_0^T \int_{\mathbb{S}^2} \int_0^\infty r^2 f(r) \partial_r (\partial_t u)^2\,drd\omega dt \\
&\quad = \int_0^T \int_{\mathbb{S}^2} \int_0^\infty r f(r) (\partial_t u)^2\,drd\omega dt + \frac{1}{2} \int_0^T \int_{\mathbb{S}^2} \int_0^\infty r^2 f'(r) (\partial_t u)^2\,drd\omega dt \\
&\quad = \int_0^T \int_{\mathbb{R}^3} \frac{f(r)}{r} (\partial_t u)^2\,dxdt + \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} f'(r) (\partial_t u)^2\,dxdt.
\end{aligned}$$

Now we have for the first part of the calculation that

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} \partial_t^2 u \left[f(r)\partial_r u + \frac{f(r)}{r} u\right] dxdt = \\
&\qquad \int_{\mathbb{R}^3} f(r)\partial_t u \partial_r u\,dx\Big|_0^T + \int_{\mathbb{R}^3} \frac{f(r)}{r} u \partial_t u\,dx\Big|_0^T + \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} f'(r) (\partial_t u)^2\,dxdt.
\end{aligned}$$

The second term of (4) is

$$\begin{aligned}
&- \sum_{i=1}^3 \int_0^T \int_{\mathbb{R}^3} \partial_{x_i}^2 u\, f(r) \left[\partial_r u + \frac{1}{r} u\right] dxdt \\
&\qquad = \sum_{i=1}^3 \int_0^T \int_{\mathbb{R}^3} \partial_{x_i} u \left[\partial_{x_i}(f(r)\partial_r u) + \partial_{x_i}\left(\frac{f(r)}{r} u\right)\right] dxdt.
\end{aligned}$$

Using the fact that $\partial_{x_i} r = \frac{x_i}{r}$, we obtain

$$= \sum_{i=1}^{3} \int_0^T \int_{\mathbb{R}^3} \partial_{x_i} u \left[f'(r) \frac{x_i}{r} \partial_r u + f(r) \partial_{x_i} \partial_r u + f'(r) \frac{x_i}{r} \frac{1}{r} u + f(r) \frac{-1}{r^2} \frac{x_i}{r} u + \frac{f(r)}{r} \partial_{x_i} u \right] dxdt,$$

and use $[\partial_{x_i}, \partial_r] = \frac{1}{r} \not\nabla_i$ from Lemma 2.2 to then get

$$\begin{aligned}
&= \int_0^T \int_{\mathbb{R}^3} f'(r)(\partial_r u)^2 \, dxdt + \sum_{i=1}^{3} \int_0^T \int_{\mathbb{R}^3} \partial_{x_i} u f(r) \left[\partial_r \partial_{x_i} u + \frac{1}{r} \not\nabla_i u \right] dxdt \\
&\quad + \int_0^T \int_{\mathbb{R}^3} f'(r) \frac{1}{r} u \partial_r u \, dxdt + \int_0^T \int_{\mathbb{R}^3} f(r) \frac{-1}{r^2} u \partial_r u \, dxdt + \int_0^T \int_{\mathbb{R}^3} \frac{f(r)}{r} |\nabla u|^2 \, dxdt \\
&= \int_0^T \int_{\mathbb{R}^3} f'(r)(\partial_r u)^2 \, dxdt + \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} f(r) \partial_r |\nabla u|^2 \, dxdt + \int_0^T \int_{\mathbb{R}^3} \frac{f(r)}{r} \nabla u \cdot \not\nabla u \, dxdt \\
&\quad + \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} f'(r) \frac{1}{r} \partial_r u^2 \, dxdt - \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} f(r) \frac{1}{r^2} \partial_r u^2 \, dxdt + \int_0^T \int_{\mathbb{R}^3} \frac{f(r)}{r} |\nabla u|^2 \, dxdt.
\end{aligned} \tag{5}$$

The third term of this expression contains

$$\nabla u \cdot \not\nabla u = \frac{x}{r} \partial_r u \cdot \not\nabla u + |\not\nabla u|^2$$

whose first term vanishes, as is seen in (2). Therefore the term becomes

$$\int_0^T \int_{\mathbb{R}^3} \frac{f(r)}{r} |\not\nabla u|^2 \, dxdt.$$

On the second, fourth, and fifth terms of (5), we convert to spherical coordinates to perform integration by parts on ∂_r.

$$\begin{aligned}
&\frac{1}{2} \int_0^T \int_{\mathbb{R}^3} f(r) \, \partial_r |\nabla u|^2 \, dxdt \\
&\qquad = \frac{1}{2} \int_0^T \int_{\mathbb{S}^2} \int_0^\infty r^2 f(r) \partial_r |\nabla u|^2 \, drd\omega dt \\
&\qquad = - \int_0^T \int_{\mathbb{S}^2} \int_0^\infty r f(r) |\nabla u|^2 \, drd\omega dt - \frac{1}{2} \int_0^T \int_{\mathbb{S}^2} \int_0^\infty r^2 f'(r) |\nabla u|^2 \, drd\omega dt \\
&\qquad = - \int_0^T \int_{\mathbb{R}^3} \frac{f(r)}{r} |\nabla u|^2 \, dxdt - \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} f'(r) |\nabla u|^2 \, dxdt
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2} \int_0^T \int_{\mathbb{R}^3} f'(r) \frac{1}{r} \, \partial_r u^2 \, dxdt \\
&\qquad = \frac{1}{2} \int_0^T \int_{\mathbb{S}^2} \int_0^\infty r f'(r) \partial_r u^2 \, drd\omega dt \\
&\qquad = -\frac{1}{2} \int_0^T \int_{\mathbb{S}^2} \int_0^\infty f'(r) u^2 \, drd\omega dt - \frac{1}{2} \int_0^T \int_{\mathbb{S}^2} \int_0^\infty r f''(r) u^2 \, drd\omega dt \\
&\qquad = -\frac{1}{2} \int_0^T \int_{\mathbb{R}^3} \frac{f'(r)}{r^2} u^2 \, dxdt - \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} \frac{f''(r)}{r} u^2 \, dxdt
\end{aligned}$$

$$
\begin{aligned}
-\frac{1}{2}\int_0^T\int_{\mathbb{R}^3} f(r)\frac{1}{r^2}\partial_r u^2\,dxdt &= -\frac{1}{2}\int_0^T\int_{\mathbb{S}^2}\int_0^\infty f(r)\partial_r u^2\,drd\omega dt\\
&= \frac{1}{2}\int_0^T\int_{\mathbb{S}^2} f(0)u(\cdot,0)^2\,d\omega dt + \frac{1}{2}\int_0^T\int_{\mathbb{S}^2}\int_0^\infty f'(r)u^2\,drd\omega dt\\
&= \frac{1}{2}\int_0^T\int_{\mathbb{S}^2} f(0)u(\cdot,0)^2\,d\omega dt + \frac{1}{2}\int_0^T\int_{\mathbb{R}^3} \frac{f'(r)}{r^2}u^2\,dxdt
\end{aligned}
$$

When we substitute these expressions back into (5), many terms cancel. The result is

$$
\begin{aligned}
-\sum_{i=1}^3\int_0^T\int_{\mathbb{R}^3} &\partial_{x_i}^2 u\, f(r)\left[\partial_r u + \frac{1}{r}u\right]\,dxdt\\
&= \int_0^T\int_{\mathbb{R}^3} f'(r)(\partial_r u)^2\,dxdt - \frac{1}{2}\int_0^T\int_{\mathbb{R}^3} f'(r)|\nabla u|^2\,dxdt\\
&+ \int_0^T\int_{\mathbb{R}^3}\frac{f(r)}{r}|\nabla\!\!\!/ u|^2\,dxdt - \frac{1}{2}\int_0^T\int_{\mathbb{R}^3}\frac{f''(r)}{r}u^2\,dxdt + \frac{1}{2}\int_0^T\int_{\mathbb{S}^2} f(0)u(\cdot,0)^2\,d\omega dt.
\end{aligned}
$$

Now we will combine both pieces of (4) together,

$$
\begin{aligned}
&\int_{\mathbb{R}^3} f(r)\partial_t u\partial_r u\,dx\Big|_0^T + \int_{\mathbb{R}^3}\frac{f(r)}{r}u\partial_t u\,dx\Big|_0^T + \frac{1}{2}\int_0^T\int_{\mathbb{R}^3} f'(r)(\partial_t u)^2\,dxdt\\
&\quad+\int_0^T\int_{\mathbb{R}^3} f'(r)(\partial_r u)^2\,dxdt - \frac{1}{2}\int_0^T\int_{\mathbb{R}^3} f'(r)|\nabla u|^2\,dxdt + \int_0^T\int_{\mathbb{R}^3}\frac{f(r)}{r}|\nabla\!\!\!/ u|^2\,dxdt\\
&\quad-\frac{1}{2}\int_0^T\int_{\mathbb{R}^3}\frac{f''(r)}{r}u^2\,dxdt + \frac{1}{2}\int_0^T\int_{\mathbb{S}^2} f(0)u(\cdot,0)^2\,d\omega dt
\end{aligned}
$$

and use $|\nabla u|^2 = (\partial_r u)^2 + |\nabla\!\!\!/ u|^2$ to further simplify:

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^3} f(r)\partial_t u\partial_r u\,dx\Big|_0^T + \int_{\mathbb{R}^3}\frac{f(r)}{r}u\partial_t u\,dx\Big|_0^T + \frac{1}{2}\int_0^T\int_{\mathbb{R}^3} f'(r)(\partial_t u)^2\,dxdt\\
&\quad+\frac{1}{2}\int_0^T\int_{\mathbb{R}^3} f'(r)(\partial_r u)^2\,dxdt + \int_0^T\int_{\mathbb{R}^3}\left(\frac{f(r)}{r}-\frac{1}{2}f'(r)\right)|\nabla\!\!\!/ u|^2\,dxdt\\
&\quad-\frac{1}{2}\int_0^T\int_{\mathbb{R}^3}\frac{f''(r)}{r}u^2\,dxdt + \frac{1}{2}\int_0^T\int_{\mathbb{S}^2} f(0)u(\cdot,0)^2\,d\omega dt.
\end{aligned}
$$

We want to show the time boundary terms are bounded by energy, and therefore would like the function $f(r)$ to be bounded. We also want the rest of the terms to be positively signed. This requires $f(r)$ to be twice continuously differentiable, have a non-negative first derivative, a non-positive second derivative, and for $\frac{f(r)}{r} - \frac{1}{2}f'(r) \geq 0$. Assuming these qualities, we begin to manipulate the above equation into our desired inequality,

$$
\begin{aligned}
&\int_{\mathbb{R}^3} f(r)\partial_t u\partial_r u\,dx\Big|_0^T + \int_{\mathbb{R}^3}\frac{f(r)}{r}u\partial_t u\,dx\Big|_0^T + \frac{1}{2}\int_0^T\int_{\mathbb{R}^3} f'(r)(\partial_t u)^2\,dxdt\\
&+\frac{1}{2}\int_0^T\int_{\mathbb{R}^3} f'(r)(\partial_r u)^2\,dxdt + \int_0^T\int_{\mathbb{R}^3}\left(\frac{f(r)}{r}-\frac{1}{2}f'(r)\right)|\nabla\!\!\!/ u|^2\,dxdt - \frac{1}{2}\int_0^T\int_{\mathbb{R}^3}\frac{f''(r)}{r}u^2\,dxdt\\
&\qquad= -\frac{1}{2}\int_0^T\int_{\mathbb{S}^2} f(0)u(\cdot,0)^2\,d\omega dt \leq 0,
\end{aligned}
$$

which implies

$$\frac{1}{2}\int_0^T\int_{\mathbb{R}^3} f'(r)(\partial_t u)^2\,dxdt + \frac{1}{2}\int_0^T\int_{\mathbb{R}^3} f'(r)(\partial_r u)^2\,dxdt$$
$$+\int_0^T\int_{\mathbb{R}^3}\left(\frac{f(r)}{r}-\frac{1}{2}f'(r)\right)|\nabla\!\!\!/ u|^2\,dxdt - \frac{1}{2}\int_0^T\int_{\mathbb{R}^3}\frac{f''(r)}{r}u^2\,dxdt$$
$$\lesssim \int_{\mathbb{R}^3}\Big|f(r)\partial_t u\partial_r u\Big|dx\Big|_0^T + \int_{\mathbb{R}^3}\left|\frac{f(r)}{r}u\partial_t u\right|dx\Big|_0^T.$$

The function $f(r)$ is bounded and thus $f(r)$ may be dropped from the right-hand side, yielding

$$\lesssim \int_{\mathbb{R}^3}\Big|\partial_t u(T,\cdot)\partial_r u(T,\cdot)\Big|dx + \int_{\mathbb{R}^3}\Big|\partial_t u(0,\cdot)\partial_r u(0,\cdot)\Big|dx$$
$$+\int_{\mathbb{R}^3}\left|\frac{u(T,\cdot)}{r}\partial_t u(T,\cdot)\right|dx + \int_{\mathbb{R}^3}\left|\frac{u(0,\cdot)}{r}\partial_t u(0,\cdot)\right|dx. \quad (6)$$

For any arbitrary fixed time t, by the Cauchy-Schwarz inequality we have

$$\int_{\mathbb{R}^3}\Big|\partial_t u(t,\cdot)\partial_r u(t,\cdot)\Big|dx \le \left(\int_{\mathbb{R}^3}(\partial_t u(t,\cdot))^2\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}(\partial_r u(t,\cdot))^2\,dx\right)^{\frac{1}{2}}$$
$$\le \left(\int_{\mathbb{R}^3}(\partial_t u(t,\cdot))^2\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}|\nabla u(t,\cdot)|^2\,dx\right)^{\frac{1}{2}} \le \left(\int_{\mathbb{R}^3}(u'(t,\cdot))^2\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}(u'(t,\cdot))^2\,dx\right)^{\frac{1}{2}}$$
$$= E[u](t).$$

Using the Cauchy-Schwarz inequality as well as the Hardy inequality, stated below,

$$\int_{\mathbb{R}^3}\frac{u(t,\cdot)^2}{r^2}dx \lesssim \int_{\mathbb{R}^3}(\partial_r u(t,\cdot))^2dx,$$

we may show

$$\int_{\mathbb{R}^3}\left|\frac{u(t,\cdot)}{r}\partial_t u(t,\cdot)\right|dx \le \left(\int_{\mathbb{R}^3}\frac{u(t,\cdot)^2}{r^2}\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}(\partial_t u(t,\cdot))^2\,dx\right)^{\frac{1}{2}}$$
$$\le \left(\int_{\mathbb{R}^3}(\partial_r u(t,\cdot))^2\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}(\partial_t u(t,\cdot))^2\,dx\right)^{\frac{1}{2}} \le E[u](t).$$

Then (6) is bounded by energy terms, which are further bounded by initial energy due to conservation of energy,

$$\le 2\,E[u](T) + 2\,E[u](0) \lesssim E[u](0)$$

and our estimate is

$$\left\|f'(r)^{\frac{1}{2}}\partial_t u\right\|^2_{L^2L^2} + \left\|f'(r)^{\frac{1}{2}}\partial_r u\right\|^2_{L^2L^2}$$
$$+\left\|\left(\frac{f(r)}{r}-\frac{1}{2}f'(r)\right)^{\frac{1}{2}}\nabla\!\!\!/ u\right\|^2_{L^2L^2} + \left\|\left(\frac{-f''(r)}{r}\right)^{\frac{1}{2}}u\right\|^2_{L^2L^2} \lesssim E[u](0). \quad (7)$$

Let us now look at potential $f(r)$ functions to use. The traits we want are that $f(r)$ is bounded, twice continuously differentiable, has a non-negative first derivative, a non-positive second derivative, and $\frac{f(r)}{r} - \frac{1}{2}f'(r) \geq 0$.

Trivially, let $f(r) = 1$. This returns a result called the Morawetz estimate [6]:

$$\left\| r^{-\frac{1}{2}} \not\nabla u \right\|_{L^2 L^2}^2 \lesssim E[u](0).$$

For a less trivial example, we propose the function

$$f(r) = \frac{r}{r+R}$$

where R is defined on spacetime regions

$$A_R \equiv \{(t,x) : R \leq \langle x \rangle \leq 2R\}.$$

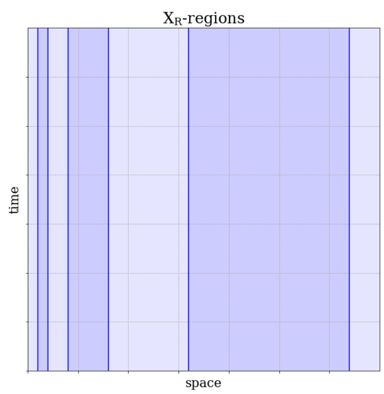

Figure 1: Shaded A_R-regions of spacetime.

The function $f(r)$ is bounded above by 1. Its first derivative

$$f'(r) = \frac{R}{(r+R)^2}$$

is positive for all $r > 0$, while its second derivative

$$f''(r) = \frac{-2R}{(r+R)^3}$$

is always negative on the same domain. To see it meets the final requirement, we calculate

$$\begin{aligned}\frac{f(r)}{r} - \frac{1}{2}f'(r) &= \frac{1}{r}\frac{r}{r+R} - \frac{1}{2}\frac{R}{(r+R)^2}\\ &= \frac{1}{r+R} - \frac{1}{2}\frac{R}{(r+R)^2}\\ &= \frac{r+\frac{1}{2}R}{(r+R)^2}\end{aligned}$$

which is always greater than 0.

In order for this estimate to give a useful result, we would like to bound $f'(r)$ and the quantity $\frac{f(r)}{r} - \frac{1}{2}f'(r)$. To do this, we examine the function on individual A_R regions. On such regions,

$$f'(r) = \frac{R}{(r+R)^2} \gtrsim \frac{R}{(R+R)^2} \gtrsim \frac{1}{R}$$

and

$$\frac{f(r)}{r} - \frac{1}{2}f'(r) = \frac{r + \frac{1}{2}R}{(r+R)^2} \geq \frac{1}{(r+R)} \gtrsim \frac{1}{R}.$$

This allows us to combine the first three terms of (7) to

$$\sup_R \left\| \frac{u'(t,x)}{\langle x \rangle^{\frac{1}{2}}} \right\|^2_{L^2L^2(A_R)}.$$

On X_R regions $\frac{-f''(r)}{r}$ is also bounded,

$$\frac{-f''(r)}{r} = \frac{2R}{r(r+R)^3} \gtrsim \frac{2R}{R(R+R)^3} \gtrsim \frac{1}{R^3}$$

yielding the final result

$$\sup_R \left\| \frac{u'(t,x)}{\langle x \rangle^{\frac{1}{2}}} \right\|^2_{L^2L^2(A_R)} + \sup_R \left\| \frac{u}{\langle x \rangle^{\frac{3}{2}}} \right\|^2_{L^2L^2(A_R)} \lesssim E[u](0).$$

■

2.4 "Ghost" Weight

For the main estimate of this paper, we will want a weight that serves a similar purpose to the $f(r)$ function above, but that is a function of $(t-r)$. Using a method introduced in [1], we construct the function

$$e^{-\sigma(t-r)}$$

where

$$\sigma(z) = \frac{z}{|z|+U}$$

and U is some constant. Then:

$$\sigma'(z) = \frac{U}{(|z|+U)^2}.$$

Because it is a function of $(t-r)$, all $\partial_t + \partial_r$ derivatives will vanish. It also has the quality

$$\partial_t \sigma(t-r) = -\partial_r \sigma(t-r).$$

Similarly to the $f(r)$-weighted estimate using restrictions to A_R-regions of spacetime, e^{σ}-weighted estimates must be restricted to certain regions of spacetime so that its derivative is bounded below. We define X_U regions of the domain by

$$\{(t,x) : U \leq \langle t-r \rangle \leq 2U\} \tag{8}$$

where $\langle x \rangle = \sqrt{1+|x|^2}$.

This allows us to bound $\sigma'(t-r)$ on X_U-regions, because

$$\sigma'(t-r) = \frac{U}{(|t-r|+U)^2} \gtrsim \frac{1}{U}.$$

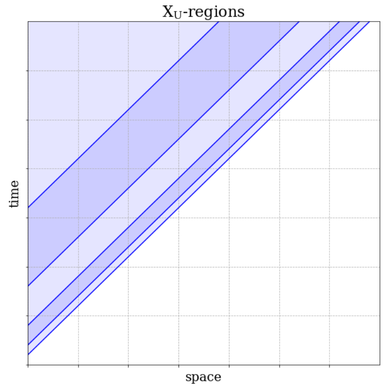

Figure 2: Shaded X_U-regions of spacetime.

3 Lemmas

The following section includes some results that will used in the main proof.

In the above calculation of the $f(r)$-weighted estimate, the integral had a multiplier of $(\partial_r + \frac{1}{r})u$. In our main estimate we will have a multiplier similar to this, which we will want to write in terms of (ru) instead of only u.

Lemma 3.1.

$$\left(\partial_t + \partial_r + \frac{1}{r}\right) u = r^{-1} \left(\partial_t + \partial_r\right)(ru)$$

Proof. We begin by evaluating $(\partial_t + \partial_r)(ru)$ using the product rule,

$$\begin{aligned}
(\partial_t + \partial_r)(ru) &= r\left(\partial_t + \partial_r\right) u + u \left(\partial_t + \partial_r\right) r \\
&= r\left(\partial_t + \partial_r\right) u + u \\
&= \left(r\partial_t + r\partial_r + 1\right) u.
\end{aligned}$$

Now we multiply both sides by r^{-1} and obtain the final result:

$$r^{-1}\left(\partial_t + \partial_r\right)(ru) = \left(\partial_t + \partial_r + \frac{1}{r}\right) u.$$

■

The next lemma is for integration by parts with the angular derivative.

Lemma 3.2. *For function $u \in C^1(\mathbb{R}_+ \times \mathbb{R}^3)$ where there exists $R > 0$ such that $u(t,x) = 0$ for all $|x| > R$,*

$$\int u \not\nabla v \, dx = -\int \not\nabla uv \, dx + \int \left(\frac{2}{r}\right) \frac{x}{r} uv \, dx.$$

Proof. Begin by examining just one component of the left side

$$\int u \not\nabla_j v \, dx = \int u \partial_{x_j} v dx - \int u \, \frac{x_j}{r} \partial_r v \, dx. \tag{9}$$

On the first term perform integration by parts. There are no boundary terms by hypothesis, and we have

$$\int u\, \partial_{x_j} v dx = -\int \partial_{x_j} u\, v\, dx.$$

On the second term, substitute $\partial_r = \sum_{i=1}^{3} \frac{x_i}{r} \partial_{x_i}$, then integrate by parts on the ∂_{x_i}. Again, there are no boundary terms.

$$\begin{aligned}
-\int u\, \frac{x_j}{r} \partial_r v\, dx &= -\sum_{i=1}^{3} \int u \frac{x_j}{r} \left(\frac{x_i}{r} \partial_{x_i} \right) v\, dx \\
&= \sum_{i=1}^{3} \int \partial_{x_i} \left(u \frac{x_j}{r} \frac{x_i}{r} \right) v\, dx \\
&= \sum_{i=1}^{3} \int \partial_{x_i} u \frac{x_j}{r} \frac{x_i}{r}\, v\, dx + \sum_{i=1}^{3} \int \partial_{x_i} \left(\frac{x_j}{r} \frac{x_i}{r} \right) u\, v\, dx \\
&= \int \frac{x_j}{r} \partial_r \left(u\right)\, v\, dx + \sum_{i=1}^{3} \int \partial_{x_i} \left(\frac{x_j x_i}{r^2} \right) u\, v\, dx.
\end{aligned}$$

Combining these two pieces, we have that (9) is equal to

$$\begin{aligned}
-\int \partial_{x_j} u\, v\, dx + \int \frac{x_j}{r} \partial_r u\, v\, dx + \sum_{i=1}^{3} \int \partial_{x_i} \left(\frac{x_j x_i}{r^2} \right) u\, v\, dx \\
= -\int \not\nabla_j (u)\, v\, dx + \sum_{i=1}^{3} \int \partial_{x_i} \left(\frac{x_j x_i}{r^2} \right) u\, v\, dx.
\end{aligned}$$

We examine the last term. Expanding with the product rule and making the substitution $\partial_{x_i} r = \frac{x_i}{r}$, we obtain

$$\begin{aligned}
\sum_{i=1}^{3} \partial_{x_i} \left(\frac{x_j x_i}{r^2} \right) &= \sum_{i=1}^{3} \left[\frac{\delta_{ij} x_i}{r^2} + \frac{x_j}{r^2} - \frac{2 x_j x_i}{r^3} \frac{x_i}{r} \right] \\
&= \frac{x_j}{r^2} + 3 \frac{x_j}{r^2} - 2 \frac{x_j}{r^2} \\
&= 2 \frac{x_j}{r^2},
\end{aligned}$$

which yields that (9) is equal to

$$\int u \not\nabla_j v\, dx = -\int \not\nabla_j u v\, dx + \int \left(\frac{2}{r} \right) \frac{x_j}{r} u v\, dx.$$

■

Corollary 3.2.1.

$$\int u \not\nabla \cdot \not\nabla v\, dx = -\int \not\nabla u \cdot \not\nabla v\, dx$$

Proof. By Lemma 3.2,

$$\int u\nabla\!\!\!/ \cdot \nabla\!\!\!/ v\,dx = -\int \nabla\!\!\!/ u \cdot \nabla\!\!\!/ v\,dx + \int \left(\frac{2}{r}u\right)\frac{x}{r}\cdot \nabla\!\!\!/ v\,dx = -\int \nabla\!\!\!/ u\cdot \nabla\!\!\!/ v\,dx$$

where the second term vanishes by the orthogonality demonstrated in (1).

■

The next lemma is referred to as "bootstrapping."

Lemma 3.3. *If $b^2 \le CA + Cab$ where C is a constant, then $b^2 \lesssim A + a^2$.*

Proof. Begin by observing the following about two numbers x and y.

$$0 \le (x-y)^2 = x^2 - 2xy + y^2$$

$$\implies xy \le \frac{1}{2}(x^2+y^2)$$

Let $b^2 \le CA + Cab$,

$$b^2 \le CA + Cab = CA + C\left(\frac{a}{\sqrt{\epsilon}}\right)(\sqrt{\epsilon}b)$$

and let $x = \left(\frac{a}{\sqrt{\epsilon}}\right)$ and $y = (\sqrt{\epsilon}b)$. Then

$$b^2 \le CA + \frac{C}{2}\left(\frac{a^2}{\epsilon} + \epsilon b^2\right)$$

$$b^2\left(1 - \frac{C}{2}\epsilon\right) \le CA + \frac{C}{2\epsilon}a^2$$

Fix $\epsilon = \frac{1}{C}$. Then

$$b^2 \le 2CA + C^2a^2$$

and we conclude the proof

$$b^2 \lesssim A + a^2.$$

■

Finally, we prove Lemma 2.2 of [5].

Lemma 3.4. *For a function $u \in C^2(\mathbb{R}_+ \times \mathbb{R}^3)$, where for all $T > 0$ there exists some $R > 0$ such that $|u(t,x)| = 0$ for all $t \in [0,T)$ and $|x| > R$, and for constant p such that $0 < p < 2$,*

$$\left\|\langle r\rangle^{\frac{p-1}{2}} r^{-1}u\right\|^2_{L^2L^2} \lesssim \left\|\langle r\rangle^{\frac{p-1}{2}} r^{-\frac{1}{2}}u(0,\cdot)\right\|^2_{L^2} + \left\|\langle r\rangle^{\frac{p-1}{2}} r^{-1}(\partial_t+\partial_r)(ru)\right\|^2_{L^2L^2}.$$

Proof.

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^3} \frac{(1+r)^{p-1}}{r^2}u^2\,dxdt &= -\int_0^T\int_{\mathbb{R}^3}(1+r)^{p-1}(\partial_t+\partial_r)\left(\frac{1}{r}\right)u^2\,dxdt\\ &= -\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^{p-1}(\partial_t+\partial_r)\left(\frac{1}{r}\right)(ru)^2\,drd\omega dt\end{aligned}$$

We perform integration by parts on $(\partial_t + \partial_r)$,

$$= -\int_{\mathbb{S}^2}\int_0^\infty \frac{(1+r)^{p-1}}{r}(ru)^2\,drd\omega\Big|_0^T + (p-1)\int_0^T\int_{\mathbb{S}^2}\int_0^\infty \frac{(1+r)^{p-2}}{r}(ru)^2\,drd\omega dt$$
$$+ 2\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^{p-1}u\,(\partial_t+\partial_r)(ru)\,drd\omega dt$$

and rearrange terms to yield

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^3}\frac{(1+r)^{p-1}}{r^2}u^2\,dxdt + \int_{\mathbb{R}^3}\frac{(1+r)^{p-1}}{r}(u(T,\cdot))^2\,dx \\ = \int_{\mathbb{R}^3}\frac{(1+r)^{p-1}}{r}(u(0,\cdot))^2\,dx + (p-1)\int_0^T\int_{\mathbb{R}^3}\frac{(1+r)^{p-2}}{r}(u)^2\,dxdt \\ + 2\int_0^T\int_{\mathbb{R}^3}\frac{(1+r)^{p-1}}{r^2}u\,(\partial_t+\partial_r)(ru)\,dxdt. \quad (10)\end{aligned}$$

The second term is positive and will be dropped from the left-hand side. On the last term we use the Cauchy-Schwarz inequality to show

$$\begin{aligned}2\int_0^T\int_{\mathbb{R}^3}\frac{(1+r)^{p-1}}{r^2}u\,(\partial_t+\partial_r)(ru)\,dxdt \\ \lesssim 2\left(\int_0^T\int_{\mathbb{R}^3}\frac{(1+r)^{p-1}}{r^2}u^2\,dxdt\right)^{\frac{1}{2}}\left(\int_0^T\int_{\mathbb{R}^3}\frac{(1+r)^{p-1}}{r^2}\left((\partial_t+\partial_r)(ru)\right)^2\,dxdt\right)^{\frac{1}{2}}.\end{aligned}$$

The first factor in the right-hand side resembles the left-hand side of (10), therefore we use Lemma 3.3 to "bootstrap" it. Finally we use the fact that $\frac{1}{1+r} \leq \frac{1}{r}$ to show the second to last term of (10) is bounded,

$$(p-1)\int_0^T\int_{\mathbb{R}^3}\frac{(1+r)^{p-2}}{r}(u)^2\,dxdt \leq |p-1|\int_0^T\int_{\mathbb{R}^3}\frac{(1+r)^{p-1}}{r^2}(u)^2\,dxdt$$

and can be subtracted over to the left-hand side. This gives the final result

$$\begin{aligned}(1-|p-1|)\int_0^T\int_{\mathbb{R}^3}\frac{(1+r)^{p-1}}{r^2}u^2\,dxdt \\ \lesssim \int_{\mathbb{R}^3}\frac{(1+r)^{p-1}}{r}(u(0,\cdot))^2\,dx + +\int_0^T\int_{\mathbb{R}^3}\frac{(1+r)^{p-1}}{r^2}\left((\partial_t+\partial_r)(ru)\right)^2\,dxdt\end{aligned}$$

where by the restriction on p the coefficient $(1-|p-1|)$ is some positive number and can be dropped.

■

4 Main Estimate

We will now prove main estimate of the paper, Theorem 1.1.

In the case of no damping, $a = 0$, we will recover the previously proven estimate [5, 3], and in the case of damping, $a > 0$, we see an improved bound.

Proof. We start by evaluating the integral

$$\int_0^T \int_{\mathbb{R}^3} (1+r)^p e^{-\sigma(t-r)} \left(\Box + a\partial_t\right) u \left(\partial_t + \partial_r + \frac{1}{r}\right) u \, dxdt.$$

In Section 2.3, we proved an estimate with a multiplier of the function $f(r)$, then determined which features this function should have. Here the $(1+r)^p e^{-\sigma(t-r)}$ serves a similar purpose. The r^p-weight method was introduced in [2], while the e^{σ} "ghost weight" method is from [1]. The $\partial_t + \partial_r$ is a "good derivative" multiplier, that is, it takes advantage of wave decay in the $t+r$ direction, and the $\frac{1}{r}$ is to cancel out some other terms. Just as in the $f(r)$-weighted estimate calculation, our goal will be to put each term into a form which has a known sign.

To begin, we will rewrite the integral in terms of (ru) using Lemmas 2.1 and 3.1:

$$= \int_0^T \int_{\mathbb{R}^3} (1+r)^p e^{-\sigma(t-r)} \frac{1}{r} \left(\partial_t^2 - \partial_r^2 - \not\nabla \cdot \not\nabla + a\partial_t\right) (ru) \frac{1}{r} \left(\partial_t + \partial_r\right) (ru) \, dxdt.$$

Then we convert to spherical coordinates, cancel the factor of r^{-2}, and expand the expression into multiple terms:

$$\begin{aligned} &= \int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^p e^{-\sigma(t-r)} \left(\partial_t^2 - \partial_r^2\right) (ru) \left(\partial_t + \partial_r\right) (ru) \, drd\omega dt \\ &\quad - \int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^p e^{-\sigma(t-r)} \left(\not\nabla \cdot \not\nabla\right) (ru) \left(\partial_t + \partial_r\right) (ru) \, drd\omega dt \\ &\quad + \int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^p e^{-\sigma(t-r)} \left(a\partial_t\right) (ru) \left(\partial_t + \partial_r\right) (ru) \, drd\omega dt. \end{aligned} \tag{11}$$

In the first term of (11), $(\partial_t^2 - \partial_r^2)$ can be factored.

$$\begin{aligned} &\int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^p \, e^{-\sigma(t-r)} \left(\partial_t^2 - \partial_r^2\right) (ru) \left(\partial_t + \partial_r\right) (ru) \, dr \, d\omega \, dt \\ &\qquad = \int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^p \, e^{-\sigma(t-r)} \left(\partial_t - \partial_r\right) \left(\partial_t + \partial_r\right) (ru) \left(\partial_t + \partial_r\right) (ru) \, dr \, d\omega \, dt. \end{aligned}$$

Then we rewrite using the chain rule relation $v \, \mathrm{d}v = \frac{1}{2} \mathrm{d}(v)^2$,

$$= \frac{1}{2} \int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^p \, e^{-\sigma(t-r)} \left(\partial_t - \partial_r\right) \left(\left(\partial_t + \partial_r\right) (ru)\right)^2 \, dr \, d\omega \, dt$$

and integrate by parts on $(\partial_t - \partial r)$ to obtain

$$\begin{aligned} =&\frac{1}{2} \int_{\mathbb{S}^2} \int_0^\infty (1+r)^p \, e^{-\sigma(t-r)} \left(\left(\partial_t + \partial_r\right) (ru)\right)^2 \, dr \, d\omega \Big|_0^T \\ &+ \int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^p \, e^{-\sigma(t-r)} \sigma'(t-r) \left(\left(\partial_t + \partial_r\right) (ru)\right)^2 \, dr \, d\omega \, dt \\ &- \frac{1}{2} \int_0^T \int_{\mathbb{S}^2} (1+r)^p \, e^{-\sigma(t-r)} \left(\left(\partial_t + \partial_r\right) (ru)\right)^2 \, d\omega \, dt \Big|_0^\infty \\ &+ \frac{p}{2} \int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^{p-1} \, e^{-\sigma(t-r)} \left(\left(\partial_t + \partial_r\right) (ru)\right)^2 \, dr \, d\omega \, dt. \end{aligned}$$

On the radial boundary term, we expand the squared quantity

$$((\partial_t + \partial_r)(ru))^2 = u^2 + 2ru(\partial_t + \partial_r)u + (r(\partial_t + \partial_r)u)^2$$

which leaves only $u(t,0)^2$ when $r = 0$:

$$=\frac{1}{2}\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}((\partial_t+\partial_r)(ru))^2\, dr\, d\omega\Big|_0^T \tag{12}$$

$$+\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\sigma'(t-r)((\partial_t+\partial_r)(ru))^2\, dr\, d\omega\, dt \tag{13}$$

$$+\frac{1}{2}\int_0^T\int_{\mathbb{S}^2} e^{-\sigma(t)}(u(t,0))^2\, d\omega\, dt$$

$$+\frac{p}{2}\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^{p-1} e^{-\sigma(t-r)}((\partial_t+\partial_r)(ru))^2\, dr\, d\omega\, dt.$$

These terms are all positively signed, so we will move on to the second term in (11):

$$-\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\left(\nabla\!\!\!/ \cdot \nabla\!\!\!/(ru)\right)(\partial_t+\partial_r)(ru)\, dr\, d\omega\, dt.$$

We integrate by parts with the angular derivative, using Corollary 3.2.1,

$$=\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\nabla\!\!\!/((\partial_t+\partial_r)(ru))\cdot\nabla\!\!\!/(ru)\, dr\, d\omega\, dt$$

and then use the commutator result $\left[\nabla\!\!\!/, \partial_t+\partial_r\right] = \frac{1}{r}\nabla\!\!\!/$ from Lemma 2.2 to obtain

$$=\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\left((\partial_t+\partial_r)\nabla\!\!\!/(ru)\right)\cdot\nabla\!\!\!/(ru)\, dr\, d\omega\, dt$$

$$+\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p r^{-1} e^{-\sigma(t-r)}|\nabla\!\!\!/(ru)|^2\, dr\, d\omega\, dt.$$

We rewrite the first term using the chain rule,

$$=\frac{1}{2}\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}(\partial_t+\partial_r)|\nabla\!\!\!/(ru)|^2\, dr\, d\omega\, dt$$

$$+\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p r^{-1} e^{-\sigma(t-r)}|\nabla\!\!\!/(ru)|^2\, dr\, d\omega\, dt$$

then integrate by parts on $(\partial_t+\partial_r)$. The radial boundary terms are all zero, and all terms where the $(\partial_t+\partial_r)$ derivative acts on $e^{-\sigma(t-r)}$ will be zero since $(\partial_t+\partial_r)(t-r) = 0$, resulting in

$$=\frac{1}{2}\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}|\nabla\!\!\!/(ru)|^2\, dr\, d\omega\,\Big|_0^T$$

$$-\frac{p}{2}\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^{p-1} e^{-\sigma(t-r)}|\nabla\!\!\!/(ru)|^2\, dr\, d\omega\, dt$$

$$+\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p r^{-1} e^{-\sigma(t-r)}|\nabla\!\!\!/(ru)|^2\, dr\, d\omega\, dt.$$

The second term is negatively signed, but it can be combined with the last term and written in a different form.

$$\begin{aligned}\left[\frac{(1+r)^p}{r}-\frac{p}{2}\frac{(1+r)^p}{1+r}\right] &= \left[\frac{(1+r)^p}{r}+\left(-\frac{p}{2}\frac{(1+r)^p}{r}+\frac{p}{2}\frac{(1+r)^p}{r}\right)-\frac{p}{2}\frac{(1+r)^p}{1+r}\right] \\ &= \frac{(1+r)^p}{r}\left(1-\frac{p}{2}\right)+\frac{p}{2}\left(\frac{(1+r)^p}{r}-\frac{(1+r)^p}{1+r}\right) \\ &= \frac{(1+r)^p}{r}\left(1-\frac{p}{2}\right)+\frac{p}{2}\left(\frac{(1+r)^p(1+r)-r(1+r)^p}{r(1+r)}\right) \\ &= \frac{(1+r)^p}{r}\left(1-\frac{p}{2}\right)+\frac{p}{2}\left(\frac{(1+r)^{p-1}}{r}\right).\end{aligned}$$

This yields terms which are all positively signed, provided that $p<2$, and the second term of (11) becomes

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p\, e^{-\sigma(t-r)}\, |\nabla\!\!\!/(ru)|^2\, dr\, d\omega\, \Big|_0^T \\ &+\left(1-\frac{p}{2}\right)\int_0^T\int_{\mathbb{S}^2}\int_0^\infty \frac{(1+r)^p}{r}e^{-\sigma(t-r)}|\nabla\!\!\!/(ru)|^2\ dr\, d\omega\, dt \\ &+\frac{p}{2}\int_0^T\int_{\mathbb{S}^2}\int_0^\infty \frac{(1+r)^{p-1}}{r}e^{-\sigma(t-r)}|\nabla\!\!\!/(ru)|^2\ dr\, d\omega\, dt.\end{aligned}$$

Now we evaluate the third term of (11), which has the damping term $a\partial_t$,

$$\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)} a\partial_t(ru)\left(\partial_t+\partial_r\right)(ru)\, dr d\omega dt.$$

We will rewrite ∂_t in the expanded form

$$\partial_t=\frac{1}{2}\left(\partial_t+\partial_r\right)+\frac{1}{2}\left(\partial_t-\partial_r\right),$$

which yields

$$\begin{aligned}&=\frac{a}{2}\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\left(\left(\partial_t+\partial_r\right)(ru)\right)^2\ dr d\omega dt \\ &\qquad +\frac{a}{2}\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\left(\partial_t-\partial_r\right)(ru)\left(\partial_t+\partial_r\right)(ru)\, dr d\omega dt.\end{aligned}$$

On the second term, we integrate by parts on $(\partial_t - \partial_r)$,

$$
\begin{aligned}
=&\frac{a}{2}\int_0^T \int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)} \left((\partial_t+\partial_r)(ru)\right)^2 \, dr d\omega dt \\
&+ \frac{a}{2}\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}(ru)\,(\partial_t+\partial_r)(ru)\, dr d\omega\Big|_0^T \\
&- \frac{a}{2}\int_0^T \int_{\mathbb{S}^2}\int_0^\infty (\partial_t-\partial_r)\left[(1+r)^p e^{-\sigma(t-r)}\,(\partial_t+\partial_r)(ru)\right](ru)\, dr d\omega dt \\
=&\frac{a}{2}\int_0^T \int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)} \left((\partial_t+\partial_r)(ru)\right)^2 \, dr d\omega dt \\
&+ \frac{a}{2}\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}(ru)\,(\partial_t+\partial_r)(ru)\, dr d\omega\Big|_0^T \\
&+ a\int_0^T \int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\sigma'(t-r)\,(\partial_t+\partial_r)(ru)(ru)\, dr d\omega dt \\
&+ p\frac{a}{2}\int_0^T \int_{\mathbb{S}^2}\int_0^\infty (1+r)^{p-1} e^{-\sigma(t-r)}\,(\partial_t+\partial_r)(ru)(ru)\, dr d\omega dt \qquad (14)\\
&- \frac{a}{2}\int_0^T \int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\left(\partial_t^2-\partial_r^2\right)(ru)(ru)\, dr d\omega dt. \qquad (15)
\end{aligned}
$$

The second and third term of this expression are not clearly signed; we will fix this later. First we will evaluate the term labeled by (14) by rewriting it with the chain rule, then integrating by parts on $(\partial_t + \partial_r)$.

$$
\begin{aligned}
p\frac{a}{2}&\int_0^T \int_{\mathbb{S}^2}\int_0^\infty (1+r)^{p-1} e^{-\sigma(t-r)}\,(\partial_t+\partial_r)(ru)(ru)\, dr d\omega dt \\
&=p\frac{a}{4}\int_0^T \int_{\mathbb{S}^2}\int_0^\infty (1+r)^{p-1} e^{-\sigma(t-r)}\,(\partial_t+\partial_r)\left((ru)^2\right) dr d\omega dt \\
&=p\frac{a}{4}\int_{\mathbb{S}^2}\int_0^\infty (1+r)^{p-1} e^{-\sigma(t-r)}(ru)^2\, dr d\omega\Big|_0^T \\
&\quad + p\frac{a}{4}(1-p)\int_0^T \int_{\mathbb{S}^2}\int_0^\infty (1+r)^{p-2} e^{-\sigma(t-r)}(ru)^2\, dr d\omega dt.
\end{aligned}
$$

On the term labeled by (15), we put this in terms of the damped wave operator using Lemma 2.1,

$$
\begin{aligned}
\left(\partial_t^2-\partial_r^2\right)(ru) &= \left(\partial_t^2-\partial_r^2-\nabla\!\!\!/\cdot\nabla\!\!\!/+a\partial_t\right)(ru) + \nabla\!\!\!/\cdot\nabla\!\!\!/(ru) - a\partial_t(ru) \\
&= r\left(\Box + a\partial_t\right)u + \nabla\!\!\!/\cdot\nabla\!\!\!/(ru) - a\partial_t(ru)
\end{aligned}
$$

and (15) becomes

$$
\begin{aligned}
&-\frac{a}{2}\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\left(r\left(\Box + a\partial_t\right)u\right)(ru)\,drd\omega dt\\
&-\frac{a}{2}\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\left(\not{\nabla}\cdot\not{\nabla}\right)(ru)(ru)\,drd\omega dt\\
&+\frac{a^2}{2}\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\partial_t(ru)(ru)\,drd\omega dt.
\end{aligned}
$$

On the second term of this we integrate by parts with the angular derivative, and on the third term we rewrite it with the chain rule and then integrate parts on ∂_t to obtain

$$
\begin{aligned}
=&-\frac{a}{2}\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\left(r\left(\Box + a\partial_t\right)u\right)(ru)\,drd\omega dt\\
&+\frac{a}{2}\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}|\not{\nabla}(ru)|^2\,drd\omega dt\\
&+\frac{a^2}{4}\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}(ru)^2\,drd\omega\Big|_0^T\\
&+\frac{a^2}{4}\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\sigma'(t-r)(ru)^2\,drd\omega dt.
\end{aligned}
$$

Substituting the corresponding expanded pieces back into (14) and (15), then the entire damped term is

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\left(a\partial_t\right)(ru)\left(\partial_t+\partial_r\right)(ru)\,drd\omega dt\\
&\quad=\frac{a}{2}\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\left(\left(\partial_t+\partial_r\right)(ru)\right)^2\,drd\omega dt\\
&\quad+\frac{a}{2}\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}(ru)\left(\left(\partial_t+\partial_r\right)(ru)\right)\,drd\omega\Big|_0^T \qquad (16)\\
&\quad+a\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\sigma'(t-r)\left(\partial_t+\partial_r\right)(ru)(ru)\,drd\omega dt \qquad (17)\\
&\quad+p\frac{a}{4}\int_{\mathbb{S}^2}\int_0^\infty (1+r)^{p-1} e^{-\sigma(t-r)}(ru)^2\,drd\omega\Big|_0^T\\
&\quad+p\frac{a}{4}(1-p)\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^{p-2} e^{-\sigma(t-r)}(ru)^2\,drd\omega dt\\
&\quad-\frac{a}{2}\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\left(r\left(\Box + a\partial_t\right)u\right)(ru)\,drd\omega dt\\
&\quad+\frac{a}{2}\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}|\not{\nabla}(ru)|^2\,drd\omega dt\\
&\quad+\frac{a^2}{4}\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}(ru)^2\,drd\omega\Big|_0^T \qquad (18)\\
&\quad+\frac{a^2}{4}\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\sigma'(t-r)(ru)^2\,drd\omega dt. \qquad (19)
\end{aligned}
$$

As mentioned earlier, terms (16) and (17) are not clearly signed. To put them in a better form, we will combine them together with other terms and then complete the square.

Term (16) is combined with (12) and (18)

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\left[((\partial_t+\partial_r)(ru))^2 + a(ru)\left((\partial_t+\partial_r)(ru)\right) + \frac{a^2}{2}(ru)^2\right] drd\omega\Big|_0^T \\
&\qquad = \frac{1}{2}\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\left[\Big(((\partial_t+\partial_r)(ru)) + \frac{a}{2}(ru)\Big)^2 + \frac{a^2}{4}(ru)^2\right] drd\omega\Big|_0^T,
\end{aligned}$$

while term (17) is combined with (13) and (19)

$$\begin{aligned}
&\int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\sigma'(t-r) \\
&\qquad\qquad \times\left[((\partial_t+\partial_r)(ru))^2 + a(ru)\left((\partial_t+\partial_r)(ru)\right) + \frac{a^2}{4}(ru)^2\right] drd\omega dt \\
&\quad = \int_0^T\int_{\mathbb{S}^2}\int_0^\infty (1+r)^p e^{-\sigma(t-r)}\sigma'(t-r)\left[\Big(((\partial_t+\partial_r)(ru)) + \frac{a}{2}(ru)\Big)^2\right] drd\omega dt.
\end{aligned}$$

Now all terms, excluding those that contain the wave operator $\Box$, are clearly signed. Combining all calculations back into one equality yields

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} (1+r)^p e^{-\sigma(t-r)} \left(\Box + a\partial_t\right) u \left(\partial_t + \partial_r + \frac{1}{r}\right) u \, dxdt \\
&+\frac{a}{2}\int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^p e^{-\sigma(t-r)} \left(r \left(\Box + a\partial_t\right)(u)\right)(ru)\, drd\omega dt \\
&\qquad = \frac{1}{2}\int_0^T \int_{\mathbb{S}^2} e^{-\sigma(t)} (u(t,0))^2 \, d\omega \, dt \\
&\qquad + \frac{p}{2}\int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^{p-1}\, e^{-\sigma(t-r)} \left(\left(\partial_t + \partial_r\right)(ru)\right)^2 \, dr\, d\omega\, dt \\
&\qquad + \frac{1}{2}\int_{\mathbb{S}^2} \int_0^\infty (1+r)^p \, e^{-\sigma(t-r)} \, |\nabla\!\!\!/ (ru)|^2 \, dr\, d\omega \, \Big|_0^T \\
&\qquad + \left(1 - \frac{p}{2}\right)\int_0^T \int_{\mathbb{S}^2} \int_0^\infty \frac{(1+r)^p}{r} e^{-\sigma(t-r)} |\nabla\!\!\!/ (ru)|^2 \, dr\, d\omega\, dt \\
&\qquad + \frac{p}{2}\int_0^T \int_{\mathbb{S}^2} \int_0^\infty \frac{(1+r)^{p-1}}{r} e^{-\sigma(t-r)} |\nabla\!\!\!/ (ru)|^2 \, dr\, d\omega\, dt \\
&\qquad + \frac{a}{2}\int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^p e^{-\sigma(t-r)} \left(\left(\partial_t + \partial_r\right)(ru)\right)^2 \, drd\omega dt \\
&\qquad + \frac{1}{2}\int_{\mathbb{S}^2} \int_0^\infty (1+r)^p e^{-\sigma(t-r)} \left[\left(\left((\partial_t + \partial_r)(ru)\right) + \frac{a}{2}(ru)\right)^2 + \frac{a^2}{4}(ru)^2\right] drd\omega \Big|_0^T \\
&\qquad + \int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^p e^{-\sigma(t-r)} \sigma'(t-r) \left[\left(\left((\partial_t + \partial_r)(ru)\right) + \frac{a}{2}(ru)\right)^2\right] drd\omega dt \\
&\qquad + p\frac{a}{4}\int_{\mathbb{S}^2} \int_0^\infty (1+r)^{p-1} e^{-\sigma(t-r)} (ru)^2 \, drd\omega \Big|_0^T \\
&\qquad + p\frac{a}{4}(1-p)\int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^{p-2} e^{-\sigma(t-r)} (ru)^2 \, drd\omega dt \\
&\qquad + \frac{a}{2}\int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^p e^{-\sigma(t-r)} |\nabla\!\!\!/ (ru)|^2 \, drd\omega dt.
\end{aligned}
$$

The function $e^{-\sigma}$ is bounded, so it can be removed from the integrals. The function $\sigma'(t-r)$ is bounded on X_U-regions, and so we have

$$
\sigma_U'(t-r) \gtrsim \frac{1}{\langle t-r \rangle} \simeq \frac{1}{U}
$$

on X_U-regions. For terms containing this function, we can restrict the integral to individual X_U regions, then take the supremum over U, yielding:

$$
\begin{aligned}
&\frac{1}{2}\int_0^T \int_{\mathbb{S}^2} (u(t,0))^2 \, d\omega dt + \left\| \langle r \rangle^{\frac{p-1}{2}} r^{-1} (\partial_t + \partial_r)(ru) \right\|_{L^2 L^2}^2 \\
&\qquad + \left\| \langle r \rangle^{\frac{p}{2}} \nabla\!\!\!/ u(T,\cdot) \right\|_{L^2}^2 + \left\| \langle r \rangle^{\frac{p}{2}} r^{-\frac{1}{2}} \nabla\!\!\!/ u \right\|_{L^2 L^2}^2 + \left\| \langle r \rangle^{\frac{p-1}{2}} r^{-\frac{1}{2}} \nabla\!\!\!/ u \right\|_{L^2 L^2}^2 \\
&+ a \left\| \langle r \rangle^{\frac{p}{2}} r^{-1} (\partial_t + \partial_r)(ru) \right\|_{L^2 L^2}^2 + \left\| \langle r \rangle^{\frac{p}{2}} r^{-1} \left((\partial_t + \partial_r)(ru(T,\cdot)) + \frac{a}{2}(ru(T,\cdot)) \right) \right\|_{L^2}^2 \\
&+ a^2 \left\| \langle r \rangle^{\frac{p}{2}} u(T,\cdot) \right\|_{L^2}^2 + \sup_U \left\| \langle r \rangle^{\frac{p}{2}} r^{-1} \langle t-r \rangle^{-\frac{1}{2}} \left((\partial_t + \partial_r)(ru) + \frac{a}{2}(ru) \right) \right\|_{L^2 L^2(X_U)}^2 \\
&\qquad + a \left\| \langle r \rangle^{\frac{p-1}{2}} u(T,\cdot) \right\|_{L^2}^2 + a \left\| \langle r \rangle^{\frac{p-2}{2}} u \right\|_{L^2 L^2}^2 + a \left\| \langle r \rangle^{\frac{p}{2}} \nabla\!\!\!/ u \right\|_{L^2 L^2}^2 \\
&\lesssim \left\| \langle r \rangle^{\frac{p}{2}} \nabla\!\!\!/ u(0,\cdot) \right\|_{L^2}^2 + \left\| \langle r \rangle^{\frac{p}{2}} r^{-1} \left((\partial_t + \partial_r)(ru(0,\cdot)) + \frac{a}{2}(ru(0,\cdot)) \right) \right\|_{L^2}^2 \\
&\qquad + a^2 \left\| \langle r \rangle^{\frac{p}{2}} u(0,\cdot) \right\|_{L^2}^2 + a \left\| \langle r \rangle^{\frac{p-1}{2}} u(0,\cdot) \right\|_{L^2}^2 \\
&\qquad\qquad + \left\| \langle r \rangle^{p} r^{-1} ((\Box + a\partial_t) u) \left[(\partial_t + \partial_r)(ru) + \frac{a}{2}(ru) \right] \right\|_{L^1 L^1}.
\end{aligned}
$$

Terms on the left-hand side can be easily removed, since subtracting them only decreases the left-hand side. The first term is not helpful, so it is dropped. The fourth and fifth terms both contain $\nabla\!\!\!/ u$, but the fourth provides a larger weight; therefore the fifth term is not helpful and is dropped as well.

The right-hand side is an upper bound, and thus will be greater than the supremum over time of the left side. Therefore all terms containing $u(T,\cdot)$ can be increased to L^∞-norms.

This leaves the final term to be rewritten such that the right-hand side contains a term with only a $(\Box + a\partial_t)$ derivative on u. The Cauchy-Schwarz inequality implies

$$
\begin{aligned}
&\left\| \langle r \rangle^{p} r^{-1} ((\Box + a\partial_t) u) \left[(\partial_t + \partial_r)(ru) + \frac{a}{2}(ru) \right] \right\|_{L^1 L^1} \\
&\qquad \lesssim \left(\sum_U \left\| \langle r \rangle^{\frac{p}{2}} \langle t-r \rangle^{\frac{1}{2}} ((\Box + a\partial_t) u) \right\|_{L^2 L^2(X_U)} \right) \\
&\qquad\qquad \left(\sup_U \left\| \langle r \rangle^{\frac{p}{2}} r^{-1} \langle t-r \rangle^{-\frac{1}{2}} \left[(\partial_t + \partial_r) u + \frac{a}{2}(ru) \right] \right\|_{L^2 L^2(X_U)} \right).
\end{aligned}
$$

The right-hand side of this expression has a factor which looks like a term in the left-hand side. Therefore we will "bootstrap" it by Lemma 3.3 to give the final result:

$$
\begin{aligned}
&\left\|\langle r\rangle^{\frac{p-1}{2}} r^{-1}(\partial_t+\partial_r)(ru)\right\|_{L^2L^2}^2+\left\|\langle r\rangle^{\frac{p}{2}}\not\nabla u\right\|_{L^\infty L^2}^2+\left\|\langle r\rangle^{\frac{p}{2}} r^{-\frac{1}{2}}\not\nabla u\right\|_{L^2L^2}^2\\
&\quad+a\left\|\langle r\rangle^{\frac{p}{2}} r^{-1}(\partial_t+\partial_r)(ru)\right\|_{L^2L^2}^2+\left\|\langle r\rangle^{\frac{p}{2}} r^{-1}\left((\partial_t+\partial_r)(ru)+\frac{a}{2}(ru)\right)\right\|_{L^\infty L^2}^2\\
&\quad+a^2\left\|\langle r\rangle^{\frac{p}{2}} u\right\|_{L^\infty L^2}^2+\sup_U\left\|\langle r\rangle^{\frac{p}{2}} r^{-1}\langle t-r\rangle^{-\frac{1}{2}}\left((\partial_t+\partial_r(ru)+\frac{a}{2}(ru)\right)\right\|_{L^2L^2(X_U)}^2\\
&\quad+a\left\|\langle r\rangle^{\frac{p-1}{2}} u\right\|_{L^\infty L^2}^2+a\left\|\langle r\rangle^{\frac{p-2}{2}} u\right\|_{L^2L^2}^2+a\left\|\langle r\rangle^{\frac{p}{2}}\not\nabla u\right\|_{L^2L^2}^2\\
&\lesssim\left\|\langle r\rangle^{\frac{p}{2}}\not\nabla u(0,\cdot)\right\|_{L^2}^2+\left\|\langle r\rangle^{\frac{p}{2}} r^{-1}\left((\partial_t+\partial_r)(ru(0,\cdot))+\frac{a}{2}(ru(0,\cdot))\right)\right\|_{L^2}^2\\
&\quad+a^2\left\|\langle r\rangle^{\frac{p}{2}} u(0,\cdot)\right\|_{L^2}^2+a\left\|\langle r\rangle^{\frac{p-1}{2}} u(0,\cdot)\right\|_{L^2}^2+\left(\sum_U U^{\frac{1}{2}}\left\|\langle r\rangle^{\frac{p}{2}}((\Box+a\partial_t)u)\right\|_{L^2L^2(X_U)}\right)^2.
\end{aligned}
$$

■

5 Discussion

In the case of $a = 0$, with no damping, we recover the original estimate from [5].

$$
\begin{aligned}
&\left\|\langle r\rangle^{\frac{p-1}{2}} r^{-1}(\partial_t+\partial_r)(ru)\right\|_{L^2L^2}^2+\left\|\langle r\rangle^{\frac{p}{2}}\not\nabla u\right\|_{L^\infty L^2}^2+\left\|\langle r\rangle^{\frac{p}{2}} r^{-\frac{1}{2}}\not\nabla u\right\|_{L^2L^2}^2\\
&\quad+\left\|\langle r\rangle^{\frac{p}{2}} r^{-1}(\partial_t+\partial_r)(ru)\right\|_{L^\infty L^2}^2+\sup_U\left\|\langle r\rangle^{\frac{p}{2}} r^{-1}\langle t-r\rangle^{-\frac{1}{2}}(\partial_t+\partial_r)(ru)\right\|_{L^2L^2(X_U)}^2\\
&\lesssim\left\|\langle r\rangle^{\frac{p}{2}}\not\nabla u(0,\cdot)\right\|_{L^2}^2+\left\|\langle r\rangle^{\frac{p}{2}} r^{-1}\left((\partial_t+\partial_r)(ru(0,\cdot))\right)\right\|_{L^2}^2+\left(\sum_U U^{\frac{1}{2}}\left\|\langle r\rangle^{\frac{p}{2}}\Box u\right\|_{L^2L^2(X_U)}\right)^2.
\end{aligned}
$$

In the case of $a = 1$, with damping, we will compare it to the undamped case and look for terms in the left-hand side which have a larger power of r, which give an improved lower bound.

For terms with the angular derivative, the damped estimate includes

$$\left\|\langle r\rangle^{\frac{p}{2}}\not\nabla u\right\|_{L^2L^2}^2$$

which shows improvement from

$$\left\|\langle r\rangle^{\frac{p}{2}} r^{-\frac{1}{2}}\not\nabla u\right\|_{L^2L^2}^2$$

that was present in the undamped estimate.

For terms with $(\partial_t + \partial_r)$, we see that the damped

$$\left\|\langle r\rangle^{\frac{p}{2}} r^{-1}(\partial_t+\partial_r)(ru)\right\|_{L^2L^2}^2$$

is also better than the undamped

$$\left\|\langle r\rangle^{\frac{p-1}{2}} r^{-1}(\partial_t+\partial_r)(ru)\right\|_{L^2L^2}^2.$$

To compare terms with no derivative on u, we will do a calculation to expand the above expression in terms of only u and not ru. We begin with the triangle inequality and use Lemma 3.1

$$\begin{aligned}\left\|\langle r\rangle^{\frac{p-1}{2}}(\partial_t+\partial_r)u\right\|^2_{L^2L^2} &= \left\|\langle r\rangle^{\frac{p-1}{2}}\left[\left(\partial_t+\partial_r+\frac{1}{r}\right)u-\frac{1}{r}u\right]\right\|^2_{L^2L^2}\\ &\lesssim \left\|\langle r\rangle^{\frac{p-1}{2}}\left(\partial_t+\partial_r+\frac{1}{r}\right)u\right\|^2_{L^2L^2}+\left\|\langle r\rangle^{\frac{p-1}{2}}r^{-1}u\right\|^2_{L^2L^2}\\ &= \left\|\langle r\rangle^{\frac{p-1}{2}}r^{-1}(\partial_t+\partial_r)(ru)\right\|^2_{L^2L^2}+\left\|\langle r\rangle^{\frac{p-1}{2}}r^{-1}u\right\|^2_{L^2L^2}.\end{aligned}$$

By adding $\left\|\langle r\rangle^{\frac{p-1}{2}}r^{-1}u\right\|^2_{L^2L^2}$ to both sides of the inequality

$$\begin{aligned}&\left\|\langle r\rangle^{\frac{p-1}{2}}(\partial_t+\partial_r)u\right\|^2_{L^2L^2}+\left\|\langle r\rangle^{\frac{p-1}{2}}r^{-1}u\right\|^2_{L^2L^2}\\ &\qquad\lesssim \left\|\langle r\rangle^{\frac{p-1}{2}}r^{-1}(\partial_t+\partial_r)(ru)\right\|^2_{L^2L^2}+\left\|\langle r\rangle^{\frac{p-1}{2}}r^{-1}u\right\|^2_{L^2L^2},\end{aligned}$$

the result is now in a form where Lemma 3.4 can be used on the last term:

$$\begin{aligned}&\left\|\langle r\rangle^{\frac{p-1}{2}}(\partial_t+\partial_r)u\right\|^2_{L^2L^2}+\left\|\langle r\rangle^{\frac{p-1}{2}}r^{-1}u\right\|^2_{L^2L^2}\\ &\qquad\lesssim \left\|\langle r\rangle^{\frac{p-1}{2}}r^{-1}(\partial_t+\partial_r)(ru)\right\|^2_{L^2L^2}+\left\|\langle r\rangle^{\frac{p-1}{2}}r^{-\frac{1}{2}}u(0,\cdot)\right\|^2_{L^2}.\end{aligned}$$

The $t=0$ boundary term can be absorbed by other boundary terms on the right-hand side of the main estimate. Therefore in the left-hand side we replace

$$\left\|\langle r\rangle^{\frac{p-1}{2}}r^{-1}(\partial_t+\partial_r)(ru)\right\|^2_{L^2L^2}$$

by

$$\left\|\langle r\rangle^{\frac{p-1}{2}}(\partial_t+\partial_r)u\right\|^2_{L^2L^2}+\left\|\langle r\rangle^{\frac{p-1}{2}}r^{-1}u\right\|^2_{L^2L^2}$$

which allows us to more explicitly compare terms in the estimate with no derivatives on u.

Thus we see that the damped estimate contains

$$\left\|\langle r\rangle^{\frac{p-2}{2}}u\right\|^2_{L^2L^2}$$

which improves upon the undamped term

$$\left\|\langle r\rangle^{\frac{p-1}{2}}r^{-1}u\right\|^2_{L^2L^2}$$

that was just calculated.

An aspect which does not show improvement is the constant p. Previously, p was fixed between 0 and 2, but in the damped estimate we require p to be fixed between 0 and 1. This causes the range of possible powers of $\langle r\rangle$ to be decreased. It is possible that another method could recover the range of p.

These findings are in line with our expectations, as we should predict improvement for a damped system of waves. For the existence proofs in which these estimates are used, the wave systems must be shown to exhibit sufficient smallness at large times and radii in order to demonstrate that solutions exist. The purpose of the estimates is to take advantage of wave decay in these systems to demonstrate this smallness. A damping force provides an additional source of decay to the system, resisting the motion of the wave and decreasing its amplitude over time. This is why we expect the presence of a damping term to improve pre-existing estimates.

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