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ORIGINAL RESEARCH ARTICLE

Long-time Existence for Systems of Quasilinear Wave Equations

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Abstract
We consider quasilinear wave equations in $(1+3)$-dimensions where the nonlinearity $F(u, u', u'')$ is permitted to depend on the solution rather than just its derivatives. For scalar equations, if $(\partial_u^2 F)(0,0,0) = 0$, almost global existence was established by Lindblad. We seek to show a related almost global existence result for coupled systems of such equations. To do so, we will rely upon a variant of the r^p-weighted local energy estimate of Dafermos and Rodnianski that includes a ghost weight akin to those used by Alinhac. The decay that is needed to close the argument comes from space–time Klainerman–Sobolev type estimates from the work of Metcalfe, Tataru, and Tohaneanu.

Keywords Nonlinear · Wave equation · Almost global existence · Local energy estimate

1 Introduction

In this article, we shall examine long-time existence for systems of $(1+3)$-dimensional quasilinear wave equations with small initial data where the nonlinearity is permitted to depend on the solution rather than just its derivatives. In particular, for $\Box = \partial_t^2 - \Delta$, we shall examine

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$$\begin{cases} \Box u^I = F^I(u, u', u''), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad I = 1, 2, \ldots, M, \\ u(0, \cdot) = f, \quad \partial_t u(0, \cdot) = g. \end{cases} \tag{1.1}$$

Here $u = (u^1, \ldots, u^M)$. We use the notation $u' = \partial u = (\partial_t u, \nabla u)$ for the space–time gradient. The smooth function F vanishes to second order at the origin, and it is linear in the u'' components. Moreover, we shall assume that

$$(\partial_{u^J} \partial_{u^K} F^I)(0, 0, 0) = 0, \quad \forall\, I, J, K = 1, 2, \ldots, M, \tag{1.2}$$

which has the effect of disallowing u^2-type terms at the quadratic level for F. In [11], almost global existence, which shows that the lifespan grows exponentially as the size of the data shrinks, was proved for scalar equations. In the current article, we seek similar lower bounds on the lifespan for systems.

In the sequel, we use Einstein's summation convention. Repeated Greek letters α, β, γ are understood to sum from 0 to 3 where $x_0 = t$. Repeated lower case Roman letters i, j, k are summed from 1 to 3, and repeated upper case letters I, J, K will be summed from 1 to M.

For simplicity, we shall truncate the nonlinearity at the quadratic level:

$$\begin{aligned} F^I(u, u', u'') \; &= a_{JK}^{I,\alpha} u^J \partial_\alpha u^K + b_{JK}^{I,\alpha\beta} \partial_\alpha u^J \partial_\beta u^K \\ &\quad + A_{JK}^{I,\alpha\beta} u^K \partial_\alpha \partial_\beta u^J + B_{JK}^{I,\alpha\beta\gamma} \partial_\gamma u^K \partial_\alpha \partial_\beta u^J. \end{aligned} \tag{1.3}$$

In the small data regime, higher order terms are better behaved. The constants will be assumed to satisfy the symmetry conditions

$$A_{JK}^{I,\alpha\beta} = A_{JK}^{I,\beta\alpha} = A_{IK}^{J,\alpha\beta}, \quad B_{JK}^{I,\alpha\beta\gamma} = B_{JK}^{I,\beta\alpha\gamma} = B_{IK}^{J,\alpha\beta\gamma}. \tag{1.4}$$

Our main result is the following statement of almost global existence.

Theorem 1.1 *Suppose that $f, g \in (C_c^\infty(\mathbb{R}^3))^M$. Moreover, assume that the smooth function F vanishes to second order at the origin, satisfies (1.2), and is subject to the symmetry conditions (1.4). Then, for $N \in \mathbb{N}$ sufficiently large, there are constants $c, \varepsilon_0 > 0$ so that if*

$$\sum_{|\mu| \le N+1} \|\partial^\mu f\|_{L^2} + \sum_{|\mu| \le N} \|\partial^\mu g\|_{L^2} \le \varepsilon \tag{1.5}$$

with $\varepsilon < \varepsilon_0$, then (1.1) has a unique solution with $u \in (C^\infty([0, T_\varepsilon] \times \mathbb{R}^3))^M$ where

$$T_\varepsilon = \exp(c/\varepsilon^{\frac{1}{3}}). \tag{1.6}$$

To aid the exposition, we have restricted to the case of compactly supported initial data. Without loss of generality, we shall take

$$\operatorname{supp} f^I, \operatorname{supp} g^I \subset \{|x| \le 2\}, \quad \forall\, I = 1, 2, \ldots, M. \tag{1.7}$$

Small data in a sufficiently weighted space would also suffice. In [11], the lower bound of the lifespan was $\exp(c/\varepsilon)$. The difference in the lifespan between [11] and Theorem 1.1 is primarily due to a logarithmic loss that occurs as a part of an endpoint Hardy inequality. See Lemma 2.5. While refinements of our argument to improve the power in (1.6) are likely possible, it is not clear what the sharp power is.

Equations such as (1.1) with nonlinearity that depends on the solution rather than just its derivatives do not mesh as simply with the energy methods that are typically employed to prove long-time existence. In [7], without the hypothesis (1.2), a lower bound of $\exp(c/\varepsilon)$ was established in $(1+4)$-dimensions. For scalar equations, the additional hypothesis (1.2) was, moreover, found to be sufficient to guarantee global existence for sufficiently small data. The analogous results in $(1+3)$-dimensions appeared in [11] where the lifespan was shown to exceed c/ε^2 without (1.2) and almost global existence was provided for scalar equations.

In both [7, 11], the restriction to scalar equations is necessitated by the use of the chain rule to write $u \cdot \partial u = \frac{1}{2}\partial u^2$ interactions in divergence form. This special form, in turn, allows for easier estimation of the solution rather than only its derivatives. See, e.g., [11, Proposition 1.8]. The article [13] extended the result of [7] by establishing small data global existence for systems (1.1) subject to (1.2) in $(1+4)$-dimensions. We establish the $(1+3)$-dimensional analog here.

The principle source of decay in our proof is obtained from space–time Klainerman–Sobolev estimates as were proved in [19]. This will be paired with variants of the integrated local energy decay estimates. In [2], r^p-weighted local energy estimates, which provide improved bounds on the "good" components of the gradient: $\partial_t + \partial_r$ and $\not\nabla := \nabla - \frac{x}{r}\partial_r$, were proved. These will be combined with a ghost weight, which originates from [1]. This permits a further improvement of the bounds in the vicinity of the light cone and meshes particularly well with the space–time Klainerman–Sobolev estimates of [19].

While the proof uses the method of invariant vector fields, it will not rely on the Lorentz boosts. While Lorentz boosts are perfectly acceptable for systems such as (1.1), they can limit further extensions to, e.g., multiple speed settings, equations in exterior domains, or equations on stationary background geometries. Our proof is readily adaptable to Dirichlet wave equations exterior to, say, star-shaped obstacles. We do not include these extraneous details here. When combined with [3, 4, 16], it (largely) completes the extension of the long-time existence results of [7, 11] to exterior domains. The work on systems in [13] was also completed exterior to star-shaped obstacles. The restriction to star-shaped obstacles is likely a convenience. It is anticipated that any geometry that permits a sufficiently rapid decay of local energy would suffice. See, e.g., [5, 6].

The key aspects to the proof are to effectively bound the solution u when typical energy methods estimate ∂u and to obtain additional decay from the derivative that must be present in at least one factor of every nonlinear term. In particular, it was the former that restricted the analysis to scalar equations in preceding results. The r^p-weights and ghost weights help with both aspects. In particular, when combined with rather standard Hardy-type estimates, improved bounds on the local energy of the solution without derivatives, when compared to e.g. (2.2), result. When attempting to gain additional decay from the derivative that must appear in at least one factor of

every term of the nonlinearity, one often relies on the scaling vector field, which near the light cone gives additional decay in $t - |x|$. The ghost weight allows us to take advantage of this additional decay off of the light cone.

Since derivatives can be exchanged for extra decay using the scaling vector field and since the r^p-weighted and ghost weighted estimates allow for much larger weights for the good derivatives, which in essence provides additional decay that can be used on other factors, one can quickly become convinced that the worst possible nonlinear terms are of the form $u(\partial_t - \partial_r)u$. Moreover when all of the vector fields land on the differentiated factor, one cannot afford to lose the additional vector field that would result from using the scaling vector field to get additional decay. In this case, we move the $\partial_t - \partial_r$ using integration by parts. Within the local energy estimates, it could land on the weights or the lower order factor, which allows us to gain additional decay, in the latter case by using the scaling vector field. Additionally it could land on the multiplier, which has the basic form $\partial_t + \partial_r$ and modulo better behaved terms results in $\Box$ effectively turning these quadratic interactions into better cubic interactions.

This article is organized as follows. In the next subsection, we shall gather some notation and preliminary results that will be used frequently throughout the paper. In Sect. 2, the integrated local energy estimates will be proved. Section 3 contains our sources of decay, which are primarily space–time versions of the Klainerman–Sobolev inequality. Finally, the main theorem is proved in Sect. 4.

1.1 Notation

The vector fields that we rely on are

$$Z = (\partial_t, \partial_1, \partial_2, \partial_3, \Omega_1, \Omega_2, \Omega_3, S)$$

where

$$S = t\partial_t + r\partial_r, \quad \Omega = x \times \nabla$$

represent the scaling vector field and generators of (spatial) rotations, respectively. We will frequently rely on the orthogonal decomposition

$$\nabla = \frac{x}{r}\partial_r + \not{\nabla},$$

and we shall use $\not{\partial} = (\partial_t + \partial_r, \not{\nabla})$ as an abbreviation for the "good" derivatives. We note that

$$\not{\nabla} = -\frac{x}{r^2} \times \Omega,$$

and as such,

$$|\not{\nabla} u| \leq \frac{1}{r}|Zu|. \tag{1.8}$$

Moreover, the following commutator will be used frequently in the proofs of local energy estimates, as it was in the seminal work [21]:

$$[\nabla, \partial_r] = [\not{\nabla}, \partial_r] = \frac{1}{r}\not{\nabla}. \tag{1.9}$$

The admissible vector fields are well known to satisfy

$$[\Box, \partial] = [\Box, \Omega_j] = 0, \quad [\Box, S] = 2\Box. \tag{1.10}$$

Moreover, we have

$$[Z, \partial] \in \text{span}(\partial), \quad |[Z, \not{\partial}]u| \leq \frac{1}{r}|Zu| + |\not{\partial}u|. \tag{1.11}$$

We shall abbreviate

$$|Z^{\leq N}u| = \sum_{|\mu|\leq N} |Z^\mu u|, \quad |\partial^{\leq N}u| = \sum_{|\mu|\leq N} |\partial^\mu u|.$$

We use L^pL^q as an abbreviation for $L_t^pL_x^q([0,T]\times\mathbb{R}^3) = L^p([0,T];L^q(\mathbb{R}^3))$. In several circumstances, it will be convenient to do an inhomogeneous dyadic decomposition of $\mathbb{R}^3$, and for this purpose we denote

$$A_R = \{R < |x| < 2R\}, \quad \tilde{A}_R = \left\{\frac{7}{8}R < |x| < \frac{17}{8}R\right\} \quad \text{if } R > 1,$$

and $A_1 = \{|x| < 2\}$, $\tilde{A}_1 = \{|x| < 17/4\}$. For the standard integrated local energy estimates, we shall employ the following notations from [19, 24]:

$$\|u\|_{LE} = \sup_{j\geq 0} 2^{-j/2}\|u\|_{L^2L^2([0,T]\times A_{2^j})}, \quad \|u\|_{LE^1} = \|(\partial u, |x|^{-1}u)\|_{LE}.$$

In the proof of the local energy estimates, we will often desire a $C^1(\mathbb{R})$, bounded, nondecreasing function and for these purposes set

$$\sigma_U(z) = \frac{z}{U+|z|}, \quad U > 0.$$

In the sequel, we shall also need dyadic decompositions in $t-r$, so we set

$$\begin{aligned} X_U &= \{(t,x) \in [0,T]\times\mathbb{R}^3 : U < t-r < 2U\} \text{ for } U > 1, \\ X_1 &= \{(t,x) \in [0,T]\times\mathbb{R}^3 : |t-r| < 2\}, \end{aligned}$$

with a similar enlargement being denoted by $\tilde{X}_U$.

The estimates from [19] rely on a mixed decomposition where the cone is divided in $|x|$ away from the light cone and in $t-|x|$ near the cone. We set $C = \{r \leq t+2\}$.

Due to the simplifying assumption that the data are compactly supported, it suffices to consider only this region, though we believe that these estimates can be extended to all of $[0, T] \times \mathbb{R}^3$ in a straightforward fashion.

We first divide into dyadic intervals in time $C_\tau = \{t \in [\tau, 2\tau] \cap [0, T], r \leq t + 2\}$. Next, we shall decompose dyadically in r or $t - r$ depending on the proximity to the light cone. For $R, U > 1$, we set

$$C_\tau^R = C_\tau \cap \{R < r < 2R\}, \quad C_\tau^U = C_\tau \cap \{U < t - r < 2U\}$$

and

$$C_\tau^{R=1} = C_\tau \cap \{r < 2\}, \quad C_\tau^{U=1} = C_\tau \cap \{|t - r| < 2\}.$$

We note that

$$C_\tau = \left(\bigcup_{1 \leq R \leq \tau/4} C_\tau^R \right) \cup \left(\bigcup_{1 \leq U \leq \tau/4} C_\tau^U \right) \cup C_\tau^{\frac{\tau}{2}},$$

where

$$C_\tau^{\frac{\tau}{2}} = C_\tau \cap \left\{ t - r \geq \frac{\tau}{2} \right\} \cap \left\{ r \geq \frac{\tau}{2} \right\}.$$

We use $\tilde{C}_\tau^R, \tilde{C}_\tau^U$ to denote enlargements of these sets on both the R/U and τ scales. In the latter case, we enlarge from $[\tau, 2\tau] \cap [0, T]$ to $[(7/8)\tau, 2\tau] \cap [0, T]$. Subsequently $\tilde{\tilde{C}}_\tau^R$ and $\tilde{\tilde{C}}_\tau^U$ will indicate further enlargements. The key observation is that

$$\langle r \rangle \approx R, \quad t - r \approx \tau, \quad \text{on } C_\tau^R, \tilde{C}_\tau^R, \tilde{\tilde{C}}_\tau^R, \quad \text{with } 1 \leq R \leq \tau/4$$

and

$$r \approx \tau, \quad \langle t - r \rangle \approx U, \quad \text{on } C_\tau^U, \tilde{C}_\tau^U, \tilde{\tilde{C}}_\tau^U, \quad \text{with } 1 \leq U \leq \tau/4.$$

In this sense, the $C_\tau^{\frac{\tau}{2}}$ region can be thought of as either a C_τ^R or a C_τ^U region. Here and throughout, R, U are understood to run over dyadic values. Here we have set $\langle r \rangle$ to be a smooth function so that $\langle r \rangle \geq 3$ and $\langle r \rangle \approx r$ for $r \gg 1$. For simplicity, we could simply take $\langle r \rangle = (3 + r)$ and $\langle T \rangle = (3 + T)$.

In the sequel, we shall need cutoffs to localize to certain regions. We fix $\beta \in C^\infty(\mathbb{R})$ so that $\beta \equiv 1$ for $z < 1$ and $\beta \equiv 0$ for $z > 2$. We then set

$$\beta_{<R}(z) = \beta(z/R), \quad \beta_{>R}(z) = 1 - \beta_{<R}(z).$$

2 Local Energy Estimate

In this section, we shall collect some integrated local energy estimates, which will serve as the main linear estimate for our proof of almost global existence. The first of these is a now standard version of the original estimates of [21]. We proceed to explore a variant of this that only bounds the "good" derivatives but with much better weights. What results is a mixture of the ghost weight method of [1] (see also the related estimates in [12]) and the r^p-weighted local energy estimates of [2].

In this examination of a quasilinear problem, it will be helpful to have estimates on small, time-dependent perturbations of the flat operator $\Box$. To this end, we define

$$(\Box_h u)^I = (\partial_t^2 - \Delta)u^I - h_J^{I,\alpha\beta}(t,x)\partial_\alpha\partial_\beta u^J.$$

Here we shall assume that

$$h_J^{I,\alpha\beta} = h_J^{I,\beta\alpha} = h_I^{J,\alpha\beta}, \quad h_J^{I,\alpha\beta} \in C^1([0,T]\times\mathbb{R}^3). \tag{2.1}$$

We shall also abbreviate

$$|h| = \sum_{I,J=1}^{M}\sum_{\alpha,\beta=0}^{3}\left|h_J^{I,\alpha\beta}\right|, \quad |\partial h| = \sum_{I,J=1}^{M}\sum_{\alpha,\beta,\gamma=0}^{3}\left|\partial_\gamma h_J^{I,\alpha\beta}\right|.$$

We begin by recalling the local energy estimate for perturbations of $\Box$ that was proved in [15]. See, also, [14, 17, 18, 23], and [20].

Theorem 2.1 *Suppose that h satisfies (2.1), that $|h|$ is sufficiently small, that $u \in (C^2([0,T]\times\mathbb{R}^3))^M$, and that for all $t \in [0,T]$, $|\partial^{\le 1}u(t,x)| \to 0$ as $|x| \to \infty$. Then,*

$$\|\partial u\|_{L^\infty L^2}^2 + \|u\|_{LE^1}^2 \lesssim \|\partial u(0,\,\cdot\,)\|_{L^2}^2 + \int_0^T\int |\Box_h u|\left(|\partial u| + \frac{|u|}{r}\right)\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^T\int \left(|\partial h| + \frac{|h|}{\langle r\rangle}\right)|\partial u|\left(|\partial u| + \frac{|u|}{r}\right)\mathrm{d}x\,\mathrm{d}t. \tag{2.2}$$

The proof of the theorem follows by pairing $\Box_h u$ with the multiplier $C\partial_t u + \sigma_{2^j}(r)\partial_r u + \frac{\sigma_{2^j}(r)}{r}u$, integrating over a space–time slab, and integrating by parts.

We next consider the following variant of Theorem 2.1. It represents a combination of the ideas of [1, 2]. The former considered multipliers with principal part of the form $r^p(\partial_t + \partial_r)u$. Rather than considering associated flux terms to bound terms similar to the third term in the left side below, we shall instead modify the multiplier using a ghost weight, which originates in [1]. This more readily allows us to perform necessary manipulations to prevent a loss of regularity due to the quasilinear nature of the problem. It will also allow us to subsequently integrate by parts to control a term using ideas akin to normal forms.

Theorem 2.2 *Fix $0 < p < 2$. Suppose that h satisfies (2.1), that $u \in (C^2([0,T]\times\mathbb{R}^3))^M$, and that for all $t \in [0,T]$, $|r^{\frac{p+2}{2}}\partial^{\le 1}u(t,x)| \to 0$ as $|x| \to \infty$. Then for any $U > 0$,*

$$\begin{aligned}
&\|r^{\frac{p-2}{2}}\not\partial(ru)\|_{L^\infty L^2}^2 + \|r^{\frac{p-3}{2}}\not\partial(ru)\|_{L^2L^2}^2 + \|r^{\frac{p-2}{2}}(\sigma_U'(t-r))^{\frac12}(\partial_t+\partial_r)(ru)\|_{L^2L^2}^2\\
&\lesssim \|r^{\frac{p-2}{2}}\not\partial(ru)(0,\,\cdot\,)\|_{L^2}^2 + \sup_{t\in[0,T]}\int r^p|h||\partial u|\Big(|\partial u| + \frac{|u|}{r}\Big)\,dx\\
&\quad + \sup_{t\in[0,T]}\Big|\int_0^t\int r^{p-1}e^{-\sigma_U(t-r)}\Box_h u\cdot\Big(\partial_t+\partial_r\Big)(ru)\,dx\,dt\Big|\\
&\quad + \int_0^T\int r^{p-1}\Big(|\partial h| + \frac{|h|}{r}\Big)|\partial u||(\partial_t+\partial_r)(ru)|\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_0^T\int r^{p-1}|h||\partial u|\Big(|\not\nabla u| + \frac{|u|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_0^T\int |h|r^{p-1}\sigma_U'(t-r)|\partial u||(\partial_t+\partial_r)(ru)|\,dx\,dt\\
&\quad + \int_0^T\int r^p\Big(|(\partial_t+\partial_r)h| + \frac{|h|}{r}\Big)|\partial u|^2\,\mathrm{d}x\,\mathrm{d}t. \qquad (2.3)
\end{aligned}$$

The implicit constant is independent of both T and U.

Proof We first note that

$$\begin{aligned}
&\int_0^t\int (\Box u)^I r^p e^{-\sigma_U(t-r)}\Big(\partial_t+\partial_r+\frac{1}{r}\Big)u^I\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_0^t\int_0^\infty\int_{\mathbb{S}^2}\Big(\partial_t^2-\partial_r^2-\not\nabla\cdot\not\nabla\Big)(ru)^I\cdot r^p e^{-\sigma_U(t-r)}\Big(\partial_t+\partial_r\Big)(ru)^I\,\mathrm{d}\omega\,\mathrm{d}r\,\mathrm{d}t.
\end{aligned}$$

Using integration by parts, we see that the right side is equal to

$$\begin{aligned}
&\frac12\int_0^t\int_0^\infty\int_{\mathbb{S}^2} r^p e^{-\sigma_U(t-r)}\Big(\partial_t-\partial_r\Big)\Big|\Big(\partial_t+\partial_r\Big)(ru)\Big|^2\,\mathrm{d}\omega\,\mathrm{d}r\,\mathrm{d}t\\
&\quad + \int_0^t\int_0^\infty\int_{\mathbb{S}^2} r^p e^{-\sigma_U(t-r)}\not\nabla(ru)^I\cdot\not\nabla\Big(\partial_t+\partial_r\Big)(ru)^I\,\mathrm{d}\omega\,\mathrm{d}r\,\mathrm{d}t.
\end{aligned}$$

Relying on (1.9), we further see that this is the same as

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty \int_{\mathbb{S}^2} r^p e^{-\sigma_U(t-r)} \left|\left(\partial_t + \partial_r\right)(ru)\right|^2 \mathrm{d}\omega\,\mathrm{d}r\Big|_{t=0}^t \\
&+\frac{p}{2}\int_0^t \int_0^\infty \int_{\mathbb{S}^2} r^{p-1} e^{-\sigma_U(t-r)} \left|\left(\partial_t + \partial_r\right)(ru)\right|^2 \mathrm{d}\omega\,\mathrm{d}r\,\mathrm{d}t \\
&+\int_0^t \int_0^\infty \int_{\mathbb{S}^2} r^p \sigma_U'(t-r) e^{-\sigma_U(t-r)} \left|\left(\partial_t + \partial_r\right)(ru)\right|^2 \mathrm{d}\omega\,\mathrm{d}r\,\mathrm{d}t \\
&+\int_0^t \int_0^\infty \int_{\mathbb{S}^2} r^{p-1} e^{-\sigma_U(t-r)} |\not\nabla(ru)|^2 \,\mathrm{d}\omega\,\mathrm{d}r\,\mathrm{d}t \\
&+\frac{1}{2}\int_0^t \int_0^\infty \int_{\mathbb{S}^2} r^p e^{-\sigma_U(t-r)} \left(\partial_t + \partial_r\right) |\not\nabla(ru)|^2 \,\mathrm{d}\omega\,\mathrm{d}r\,\mathrm{d}t.
\end{aligned}$$

A final integration by parts then gives that

$$\begin{aligned}
&\int_0^t \int (\Box u)^I r^p e^{-\sigma_U(t-r)} \left(\partial_t + \partial_r + \frac{1}{r}\right) u^I \,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^\infty \int_{\mathbb{S}^2} r^p e^{-\sigma_U(t-r)} \left(\left|\left(\partial_t + \partial_r\right)(ru)\right|^2 + |\not\nabla(ru)|^2\right) \mathrm{d}\omega\,\mathrm{d}r\Big|_{t=0}^t \\
&+\frac{p}{2}\int_0^t \int_0^\infty \int_{\mathbb{S}^2} r^{p-1} e^{-\sigma_U(t-r)} \left|\left(\partial_t + \partial_r\right)(ru)\right|^2 \mathrm{d}\omega\,\mathrm{d}r\,\mathrm{d}t \\
&+\left(1-\frac{p}{2}\right)\int_0^t \int_0^\infty \int_{\mathbb{S}^2} r^{p-1} e^{-\sigma_U(t-r)} |\not\nabla(ru)|^2 \,\mathrm{d}\omega\,\mathrm{d}r\,\mathrm{d}t \\
&+\frac{1}{2}\int_0^t \int_0^\infty \int_{\mathbb{S}^2} r^p \sigma_U'(t-r) e^{-\sigma_U(t-r)} \left|\left(\partial_t + \partial_r\right)(ru)\right|^2 \mathrm{d}\omega\,\mathrm{d}r\,\mathrm{d}t. \qquad (2.4)
\end{aligned}$$

This string of equalities proves the desired estimate when $h \equiv 0$.

We now consider the perturbation. Using integration by parts, we note that

$$
\begin{aligned}
&-\int_0^t \int h_J^{I,\alpha\beta}\partial_\alpha\partial_\beta u^J \cdot r^p e^{-\sigma_U(t-r)}\Big(\partial_t+\partial_r+\frac{1}{r}\Big)u^I \,\mathrm{d}x\,\mathrm{d}t\\
&\quad=-\int h_J^{I,0\beta} r^{p-1} e^{-\sigma_U(t-r)}\partial_\beta u^J\Big(\partial_t+\partial_r\Big)(ru^I)\,\mathrm{d}x\Big|_{t=0}^t\\
&\qquad+\int_0^t\int (\partial_\alpha h_J^{I,\alpha\beta}) r^{p-1} e^{-\sigma_U(t-r)}\partial_\beta u^J\Big(\partial_t+\partial_r\Big)(ru^I)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\int_0^t\int h_J^{I,\alpha\beta} r^{-1}\partial_\alpha\Big(r^p e^{-\sigma_U(t-r)}\Big)\partial_\beta u^J\Big(\partial_t+\partial_r\Big)(ru^I)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\int_0^t\int h_J^{I,\alpha\beta} r^p e^{-\sigma_U(t-r)}\partial_\beta u^J\partial_\alpha\Big(\partial_t+\partial_r+\frac{1}{r}\Big)u^I\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Commuting the ∂_α using (1.9) and using the symmetries (2.1), we see that

$$
\begin{aligned}
&\int_0^t\int h_J^{I,\alpha\beta} r^p e^{-\sigma_U(t-r)}\partial_\beta u^J\partial_\alpha\Big(\partial_t+\partial_r+\frac{1}{r}\Big)u^I\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_0^t\int h_J^{I,k\beta} r^{p-1} e^{-\sigma_U(t-r)}\partial_\beta u^J \not\nabla_k u^I\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad-\int_0^t\int h_J^{I,k\beta} r^{p-2} e^{-\sigma_U(t-r)}\frac{x_k}{r}u^I\partial_\beta u^J\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\frac{1}{2}\int_0^t\int h_J^{I,\alpha\beta} r^p e^{-\sigma_U(t-r)}\Big(\partial_t+\partial_r+\frac{2}{r}\Big)\Big[\partial_\beta u^J\partial_\alpha u^I\Big]\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Combining the above two identities and integrating by parts, we obtain

$$
\begin{aligned}
&-\int_0^t \int h_J^{I,\alpha\beta}\partial_\alpha\partial_\beta u^J \cdot r^p e^{-\sigma_U(t-r)}\Big(\partial_t+\partial_r+\frac{1}{r}\Big)u^I \,\mathrm{d}x\,\mathrm{d}t\\
&\quad=-\int h_J^{I,0\beta} r^{p-1} e^{-\sigma_U(t-r)}\partial_\beta u^J\Big(\partial_t+\partial_r\Big)(ru^I)\,\mathrm{d}x\Big|_{t=0}^t\\
&\qquad+\frac{1}{2}\int h_J^{I,\alpha\beta} r^p e^{-\sigma_U(t-r)}\partial_\beta u^J\partial_\alpha u^I\,\mathrm{d}x\Big|_{t=0}^t\\
&\qquad+\int_0^t\int (\partial_\alpha h_J^{I,\alpha\beta}) r^{p-1} e^{-\sigma_U(t-r)}\partial_\beta u^J\Big(\partial_t+\partial_r\Big)(ru^I)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+p\int_0^t\int h_J^{I,k\beta}\frac{x_k}{r}r^{p-2}e^{-\sigma_U(t-r)}\partial_\beta u^J\Big(\partial_t+\partial_r\Big)(ru^I)\,dx\,dt\\
&+\int_0^t\int h_J^{I,k\beta}r^{p-1}e^{-\sigma_U(t-r)}\partial_\beta u^J\not{\nabla}_k u^I\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_0^t\int h_J^{I,k\beta}r^{p-2}e^{-\sigma_U(t-r)}\frac{x_k}{r}u^I\partial_\beta u^J\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^t\int h_J^{I,\alpha\beta}\omega_\alpha r^{p-1}\sigma_U'(t-r)e^{-\sigma_U(t-r)}\partial_\beta u^J(\partial_t+\partial_r)(ru^I)\,\mathrm{d}x\,\mathrm{d}t\\
&-\frac{1}{2}\int_0^t\int\Big(\partial_t+\partial_r\Big)(h_J^{I,\alpha\beta})r^pe^{-\sigma_U(t-r)}\partial_\beta u^J\partial_\alpha u^I\,\mathrm{d}x\,\mathrm{d}t\\
&-\frac{p}{2}\int_0^t\int h_J^{I,\alpha\beta}r^{p-1}e^{-\sigma_U(t-r)}\partial_\beta u^J\partial_\alpha u^I\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{2.5}
$$

Here $\omega=(-1,x/r)$. Our estimate (2.3) is an immediate consequence of (2.4) and (2.5) and taking the supremum over $t\in[0,T]$.

We next consider a Hardy-type inequality to obtain associated bounds on the solution, which are analogous to the bounds on $\||x|^{-1}u\|_{LE}$ in (2.2).

Lemma 2.3 *Let $0<p<2$. Suppose $u\in C^1([0,T]\times\mathbb{R}^3)$ and that for every $t\in[0,T]$, $|r^{p/2}u(t,x)|\to 0$ as $|x|\to\infty$. Then*

$$
\|r^{\frac{p-3}{2}}u\|_{L^2L^2}+\|r^{\frac{p-2}{2}}u\|_{L^\infty L^2}\lesssim\|r^{\frac{p-2}{2}}u(0,\,\cdot\,)\|_{L^2}+\|r^{\frac{p-3}{2}}\not{\partial}(ru)\|_{L^2L^2}. \tag{2.6}
$$

Proof By integrating by parts, for any $t\in[0,T]$, we have

$$
\begin{aligned}
&\frac{1}{2-p}\int r^{p-2}u^2(t,x)\,\mathrm{d}x+\int_0^t\int r^{p-3}u^2(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad=\frac{1}{2-p}\int r^{p-2}u^2(t,x)\,\mathrm{d}x+\frac{1}{p-2}\int_0^t\int_{\mathbb{S}^2}\int_0^\infty(\partial_\tau+\partial_r)(r^{p-2})(ru)^2\,\mathrm{d}r\,\mathrm{d}\omega\,\mathrm{d}\tau\\
&\quad=\frac{1}{2-p}\int r^{p-2}u^2(0,x)\,\mathrm{d}x-\frac{2}{p-2}\int_0^t\int_{\mathbb{S}^2}\int_0^\infty r^{p-2}(ru)(\partial_t+\partial_r)(ru)\,\mathrm{d}r\,\mathrm{d}\omega\,\mathrm{d}\tau.
\end{aligned}
$$

As the Schwarz inequality allows us to bound the last term by

$$C\left(\int_0^t\int r^{p-3}u^2\,\mathrm{d}x\,\mathrm{d}\tau\right)^{1/2}\left(\int_0^t\int r^{p-3}\left[(\partial_t+\partial_r)(ru)\right]^2\,\mathrm{d}x\,\mathrm{d}\tau\right)^{1/2}$$

and as the first factor can be bootstrapped, (2.6) follows upon taking a supremum in $t\in[0,T]$.

While we will not directly use the next lemma, which indicates the form of the lower order bound with decay in $t-r$ and $0<p<1$, we include it for the sake of completeness.

Lemma 2.4 *Let $0\le p<1$. Suppose $u\in C^1([0,T]\times\mathbb{R}^3)$ and that for every $t\in[0,T]$, $|u(t,x)|\to 0$ as $|x|\to\infty$. Then*

$$\begin{aligned}
&\int_0^T\int r^{p-2}\sigma_U'(t-r)e^{-\sigma_U(t-r)}u^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\sup_{t\in[0,T]}\int r^{p-1}\sigma_U'(t-r)e^{-\sigma_U(t-r)}u^2(t,x)\,\mathrm{d}x\\
&\lesssim\int r^{p-1}\sigma_U'(-r)e^{-\sigma_U(-r)}u^2(0,x)\,\mathrm{d}x\\
&\quad+\int_0^T\int r^{p-2}\sigma_U'(t-r)e^{-\sigma_U(t-r)}\Big[(\partial_t+\partial_r)(ru)\Big]^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{2.7}$$

Proof We argue similarly to the preceding lemma and apply integration by parts and the Schwarz inequality to observe that

$$\begin{aligned}
&\int_0^t\int r^{p-2}\sigma_U'(\tau-r)e^{-\sigma_U(\tau-r)}u^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+\frac{1}{1-p}\int r^{p-1}\sigma_U'(t-r)e^{-\sigma_U(t-r)}u^2(t,x)\,\mathrm{d}x\\
&=\frac{1}{p-1}\int_0^t\int_{\mathbb{S}^2}\int_0^\infty(\partial_\tau+\partial_r)[r^{p-1}\sigma_U'(\tau-r)e^{-\sigma_U(\tau-r)}](ru)^2\,\mathrm{d}r\,\mathrm{d}\omega\,\mathrm{d}\tau\\
&\quad+\frac{1}{1-p}\int r^{p-1}\sigma_U'(t-r)e^{-\sigma_U(t-r)}u^2(t,x)\,\mathrm{d}x\\
&=\frac{1}{1-p}\int r^{p-1}\sigma_U'(-r)e^{-\sigma_U(-r)}u^2(0,x)\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&- \frac{2}{p-1} \int_0^t \int_{\mathbb{S}^2} \int_0^\infty r^{p-1} \sigma_U'(\tau - r) e^{-\sigma_U(\tau - r)} r u (\partial_t + \partial_r)(ru) \, \mathrm{d}r \, \mathrm{d}\omega \, \mathrm{d}\tau \\
&\lesssim \int r^{p-1} \sigma_U'(-r) e^{-\sigma_U(-r)} u^2(0, x) \, \mathrm{d}x \\
&\quad + \left(\int_0^t \int r^{p-2} \sigma_U'(\tau - r) e^{-\sigma_U(\tau - r)} u^2 \, \mathrm{d}x \, \mathrm{d}\tau \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_0^t \int r^{p-2} \sigma_U'(\tau - r) e^{-\sigma_U(\tau - r)} \left[(\partial_t + \partial_r)(ru) \right]^2 \, \mathrm{d}x \, \mathrm{d}\tau \right)^{\frac{1}{2}}.
\end{aligned}
$$

We may then bootstrap the first factor of the last term and take a supremum in t to complete the argument.

We shall need the analog of the above when $p = 1$, which comes with a logarithmic loss. It is this logarithm that is largely responsible for the difference between (1.6) and the $\exp(c/\varepsilon)$ lifespan of [11].

Lemma 2.5 *Suppose $u \in C^1([0, T] \times \mathbb{R}^3)$ and that for every $t \in [0, T]$, $|u(t, x)| \to 0$ as $|x| \to \infty$. Then*

$$
\begin{aligned}
&\int_0^T \int \beta_{>2}(r) \frac{1}{r(\log\langle r \rangle)^2} \sigma_U'(t - r) e^{-\sigma_U(t - r)} u^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \sup_{t \in [0,T]} \int \beta_{>2}(r) \frac{1}{\log\langle r \rangle} \sigma_U'(t - r) e^{-\sigma_U(t - r)} u^2(t, x) \, \mathrm{d}x \\
&\lesssim \int \frac{1}{\log\langle r \rangle} \sigma_U'(-r) e^{-\sigma_U(-r)} u^2(0, x) \, \mathrm{d}x \\
&\qquad + \int_0^T \int r^{-1} \sigma_U'(t - r) e^{-\sigma_U(t - r)} \Big[\Big(\partial_t + \partial_r \Big)(ru) \Big]^2 \, \mathrm{d}x \, \mathrm{d}t + \|u\|_{LE^1}^2. \quad (2.8)
\end{aligned}
$$

Proof We observe that

$$
\begin{aligned}
&\int_0^t \int \beta_{>2}(r) \frac{1}{r(\log r)^2} \sigma_U'(\tau - r) e^{-\sigma_U(\tau - r)} u^2 \, \mathrm{d}x \, \mathrm{d}\tau \\
&\quad + \int \beta_{>2}(r) \frac{1}{\log r} \sigma_U'(t - r) e^{-\sigma_U(t - r)} u^2(t, x) \, \mathrm{d}x \\
&= \int_0^t \int_{\mathbb{S}^2} \int_0^\infty \beta_{>2}(r) \Big(\partial_\tau + \partial_r \Big) \Big[-\frac{1}{\log r} \sigma_U'(\tau - r) e^{-\sigma_U(\tau - r)} \Big] (ru)^2 \, \mathrm{d}r \, \mathrm{d}\omega \, \mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\int \beta_{>2}(r)\frac{1}{\log r}\sigma_U'(t-r)e^{-\sigma_U(t-r)}u^2(t,x)\,\mathrm{d}x\\
&=2\int_0^t\int_{\mathbb{S}^2}\int_0^\infty \beta_{>2}(r)\frac{1}{\log r}\sigma_U'(\tau-r)e^{-\sigma_U(\tau-r)}(ru)\Big(\partial_t+\partial_r\Big)(ru)\,\mathrm{d}r\,\mathrm{d}\omega\,\mathrm{d}\tau\\
&+\int \beta_{>2}(r)\frac{1}{\log r}\sigma_U'(-r)e^{-\sigma_U(-r)}u^2(0,x)\,\mathrm{d}x\\
&+\int_0^t\int \beta_{>2}'(r)\frac{1}{\log r}\sigma_U'(\tau-r)e^{-\sigma_U(\tau-r)}u^2\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}
$$

As $\operatorname{supp}\beta_{>2}'\subset[2,4]$, the last term is bounded by $\|u\|_{LE^1}^2$. The Schwarz inequality shows that the first term in the right side is

$$
\begin{aligned}
&\lesssim\left(\int_0^t\int \beta_{>2}(r)\frac{1}{r(\log r)^2}\sigma_U'(\tau-r)e^{-\sigma_U(\tau-r)}u^2\,\mathrm{d}x\,\mathrm{d}\tau\right)^{1/2}\\
&\quad\times\left(\int_0^t\int \beta_{>2}(r)r^{-1}\sigma_U'(\tau-r)e^{-\sigma_U(\tau-r)}\Big[\Big(\partial_t+\partial_r\Big)(ru)\Big]^2\,\mathrm{d}x\,\mathrm{d}\tau\right)^{1/2}.
\end{aligned}
$$

The first factor may be bootstrapped, and upon taking a supremum over $t\in[0,T]$, the proof is complete.

The following corollary combines (2.3), (2.6), and (2.7) when $p=1$ with (1.10) and provides the primary linear estimate that our proof is based upon.

Corollary 2.6 *Fix $N\in\mathbb{N}$. Suppose that h satisfies (2.1), that $u\in C^2([0,T]\times\mathbb{R}^3)$, and that for all $t\in[0,T]$, $|r^{\frac{p+2}{2}}\partial^{\le1}Z^{\le N}u(t,x)|\to0$ as $|x|\to\infty$. Then,*

$$
\begin{aligned}
&\|\langle r\rangle^{\frac12}\not\partial Z^{\le N}u\|_{L^\infty L^2}^2+\|r^{-\frac12}Z^{\le N}u\|_{L^\infty L^2}^2+\|\not\partial Z^{\le N}u\|_{L^2L^2}^2+\|r^{-1}Z^{\le N}u\|_{L^2L^2}^2\\
&\quad+\sup_{U\ge1}\Big(\|r^{-\frac12}\langle t-r\rangle^{-\frac12}(\partial_t+\partial_r)(rZ^{\le N}u)\|_{L^2L^2(X_U)}^2\\
&\quad+\|r^{-\frac12}(\log\langle r\rangle)^{-1}\langle t-r\rangle^{-\frac12}Z^{\le N}u\|_{L^2L^2(X_U)}^2\Big)\\
&\lesssim\|r^{-\frac12}(r\not\partial)^{\le1}Z^{\le N}u(0,\,\cdot\,)\|_{L^2}^2+\sup_{t\in[0,T]}\int r|h||\partial Z^{\le N}u|\Big(|\partial Z^{\le N}u|+\frac{|Z^{\le N}u|}{r}\Big)\,\mathrm{d}x\\
&\quad+\sup_{U\ge1}\sup_{t\in[0,T]}\Big|\int_0^t\int e^{-\sigma_U(t-r)}\Box_hZ^{\le N}u\cdot\Big(\partial_t+\partial_r\Big)(rZ^{\le N}u)\,\mathrm{d}x\,\mathrm{d}t\Big|\\
&\quad+\int_0^T\int\Big(|\partial h|+\frac{|h|}{r}\Big)|\partial Z^{\le N}u||(\partial_t+\partial_r)(rZ^{\le N}u)|\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^T\int |h||\partial Z^{\le N}u|\Big(|\not\nabla Z^{\le N}u|+\frac{|Z^{\le N}u|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T\int |h|\langle t-r\rangle^{-1}|\partial Z^{\le N}u||(\partial_t+\partial_r)(rZ^{\le N}u)|\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T\int r\Big(|(\partial_t+\partial_r)h|+\frac{|h|}{r}\Big)|\partial Z^{\le N}u|^2\,\mathrm{d}x\,\mathrm{d}t\\
&+\|\partial Z^{\le N}u\|_{L^\infty L^2}^2+\|Z^{\le N}u\|_{LE^1}^2. \qquad (2.9)
\end{aligned}
$$

Proof Using (2.6) and (2.8), we may adapt (2.3) with $p=1$ to the bound

$$
\begin{aligned}
&\|\langle r\rangle^{\frac12}\not\partial Z^{\le N}u\|_{L^\infty L^2}^2+\|r^{-\frac12}Z^{\le N}u\|_{L^\infty L^2}^2+\|\not\partial Z^{\le N}u\|_{L^2L^2}^2+\|r^{-1}Z^{\le N}u\|_{L^2L^2}^2\\
&\quad+\|r^{-\frac12}(\sigma_U'(t-r))^{\frac12}(\partial_t+\partial_r)(rZ^{\le N}u)\|_{L^2L^2}^2\\
&\quad+\|r^{-\frac12}(\log(\langle r\rangle)^{-1}(\sigma_U'(t-r))^{\frac12}Z^{\le N}u\|_{L^2L^2}^2\\
&\quad\lesssim \|r^{-\frac12}(r\not\partial)^{\le1}Z^{\le N}u(0,\,\cdot\,)\|_{L^2}^2+\sup_{t\in[0,T]}\int r|h||\partial Z^{\le N}u|\Big(|\partial Z^{\le N}u|+\frac{|Z^{\le N}u|}{r}\Big)\,\mathrm{d}x\\
&\qquad+\sup_{t\in[0,T]}\Big|\int_0^t\int e^{-\sigma_U(t-r)}\Box_h Z^{\le N}u\cdot\Big(\partial_t+\partial_r\Big)(rZ^{\le N}u)\,\mathrm{d}x\,\mathrm{d}t\Big|\\
&\qquad+\int_0^T\int\Big(|\partial h|+\frac{|h|}{r}\Big)|\partial Z^{\le N}u||(\partial_t+\partial_r)(rZ^{\le N}u)|\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\int_0^T\int |h||\partial Z^{\le N}u|\Big(|\not\nabla Z^{\le N}u|+\frac{|Z^{\le N}u|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\int_0^T\int |h|\sigma_U'(t-r)|\partial Z^{\le N}u||(\partial_t+\partial_r)(rZ^{\le N}u)|\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\int_0^T\int r\Big(|(\partial_t+\partial_r)h|+\frac{|h|}{r}\Big)|\partial Z^{\le N}u|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\|\partial Z^{\le N}u\|_{L^\infty L^2}^2+\|Z^{\le N}u\|_{LE^1}^2. \qquad (2.10)
\end{aligned}
$$

We note that

$$
\sigma_U'(t-r)\gtrsim\frac{1}{\langle t-r\rangle}\ \text{on } X_U,\quad\text{and}\quad \sigma_U'(t-r)\lesssim\frac{1}{\langle t-r\rangle}\ \text{provided } U\ge1.
$$

Using these facts in (2.10) and subsequently taking a supremum in U yields (2.9).

This last Hardy-type inequality is not strictly necessary. When we set up an iteration to solve (1.1), this will be a convenience when showing that the sequence converges. In particular, it will allow us to focus only on energy and integrated local energy spaces for this portion of the argument. A closely related calculation appears in [11].

Lemma 2.7 *Suppose that $u \in C^1([0,T]\times\mathbb{R}^3)$ is supported where $\{r \le t+2\}$. Then*

$$\int \frac{1}{(1+r)(t-r+3)^2}u^2\,\mathrm{d}x \lesssim \int \frac{1}{(1+r)r^2}u^2\,\mathrm{d}x + \int \frac{1}{(1+r)}(\partial_r u)^2\,\mathrm{d}x. \tag{2.11}$$

Proof For $t \in [0,T]$ fixed, we integrate by parts and apply the Schwarz inequality to obtain

$$\begin{aligned}
\int \frac{1}{(1+r)(t-r+3)^2}u^2\,\mathrm{d}x &= \int_{\mathbb{S}^2}\int_0^\infty \partial_r[(t-r+3)^{-1}]\frac{r^2}{(1+r)}u^2\,\mathrm{d}r\,\mathrm{d}\omega \\
&= -\int \frac{1}{t-r+3}\cdot\frac{2+r}{r(1+r)^2}u^2\,\mathrm{d}x \\
&\quad - 2\int \frac{1}{(1+r)(t-r+3)}u\,\partial_r u\,\mathrm{d}x \\
&\lesssim \Big(\int \frac{1}{(1+r)(t-r+3)^2}u^2\,\mathrm{d}x\Big)^{\frac12} \\
&\quad \times\Big[\Big(\int \frac{1}{r^2(1+r)}u^2\,\mathrm{d}x\Big)^{\frac12} + \Big(\int \frac{1}{(1+r)}(\partial_r u)^2\,\mathrm{d}x\Big)^{\frac12}\Big].
\end{aligned}$$

Dividing both sides by the first factor in the right completes the proof.

3 Sobolev Estimates

The main decay estimate that we shall rely upon is a space–time variant of the Klainerman–Sobolev estimate [8] that was established in [19] and is particularly well adapted to integrated local energy estimates.

As is described in Sect. 1.1, we will break space–time up into C_τ^R and C_τ^U regions where $\tau \in [0,T]$ and $1 \le R, U \le \tau/4$. On these regions, we have the following weighted Sobolev estimates, which will serve as our source of decay.

Lemma 3.1 *For any $\tau \ge 1$ and $1 \le R, U \le \tau/4$, we have*

$$\|w\|_{L^\infty L^\infty(C_\tau^R)} \lesssim \frac{1}{\tau^{\frac12}R^{\frac32}}\|Z^{\le 4}w\|_{L^2L^2(\tilde{C}_\tau^R)} + \frac{1}{\tau^{\frac12}R^{\frac12}}\|(\partial_t+\partial_r)Z^{\le 3}w\|_{L^2L^2(\tilde{C}_\tau^R)}, \tag{3.1}$$

$$\|w\|_{L^\infty L^\infty(C_\tau^U)} \lesssim \frac{1}{\tau^{\frac32}U^{\frac12}}\|Z^{\le 4}w\|_{L^2L^2(\tilde{C}_\tau^U)} + \frac{1}{\tau^{\frac32}U^{\frac12}}\|(\partial_t+\partial_r)(rZ^{\le 3}w)\|_{L^2L^2(\tilde{C}_\tau^U)}, \tag{3.2}$$

$$\|w\|_{L^\infty L^\infty(C_\tau^{\frac{\tau}{2}})} \lesssim \frac{1}{\tau^2}\|Z^{\leq 4}w\|_{L^2L^2(\tilde{C}_\tau^{\frac{\tau}{2}})} + \frac{1}{\tau}\|(\partial_t+\partial_r)Z^{\leq 3}w\|_{L^2L^2(\tilde{C}_\tau^{\frac{\tau}{2}})} \tag{3.3}$$

See [19, Lemma 3.8]. The proof of (3.1) follows by changing coordinates to $t = e^s$, $r = e^{s+\rho}$ and applying Sobolev embeddings in ω and the Fundamental Theorem of Calculus in s and ρ. In fact, this yields

$$\begin{aligned}\|w\|_{L^\infty L^\infty(C_\tau^R)} \lesssim\ & \frac{1}{\tau^{\frac{1}{2}}R^{\frac{3}{2}}}\|Z^{\leq 3}w\|_{L^2L^2(\tilde{C}_\tau^R)} \\ & + \frac{1}{\tau^{\frac{1}{2}}R}\|Z^{\leq 3}w\|^{1/2}_{L^2L^2(\tilde{C}_\tau^R)}\|\partial_r Z^{\leq 3}w\|^{1/2}_{L^2L^2(\tilde{C}_\tau^R)}\end{aligned} \tag{3.4}$$

for any $R > 1$. In order to get additional decay out of differentiated terms, such as those appearing in the last term of (3.4), the preceding work [13] in (1+4)-dimensions relied upon [19, Lemma 3.11]. As (2.9) provides better control on the good derivatives, we can argue more simply and instead use

$$(\partial_t - \partial_r) = \frac{2}{t-r}S - \frac{t+r}{t-r}(\partial_t+\partial_r). \tag{3.5}$$

As $(t+r)/(t-r) = \mathcal{O}(1)$ on $\tilde{C}_\tau^R$ with $R \leq \tau/4$, (3.1) follows immediately. Replacing w by $\beta_{>\tau/2}(t-r)w$ in (3.4) and using that $S(\beta_{>\tau/2}(t-r)) = \mathcal{O}(1)$, (3.3) is obtained similarly.

When $U = 1$, the other estimate (3.2) is an immediate corollary of (3.4) as we need only consider ∂_r as a vector field. When $U > 1$,

$$\|w\|_{L^\infty L^\infty(C_\tau^U)} \lesssim \frac{1}{\tau^{\frac{3}{2}}U^{\frac{1}{2}}}\|Z^{\leq 3}w\|_{L^2L^2(\tilde{C}_\tau^U)} + \frac{U^{\frac{1}{2}}}{\tau^{\frac{3}{2}}}\|\partial Z^{\leq 3}w\|_{L^2L^2(\tilde{C}_\tau^U)} \tag{3.6}$$

follows from arguments similar to the above in coordinates $t+r = e^s$, $t-r = e^{s+\rho}$. Subsequently applying (3.5) yields (3.2).

It will also be helpful to have the following common weighted Sobolev estimate of [9].

Lemma 3.2 *Provided that $h \in C^\infty(\mathbb{R}^3)$,*

$$\|h\|_{L^\infty(A_R)} \lesssim R^{-1}\|Z^{\leq 2}h\|_{L^2(\tilde{A}_R)}. \tag{3.7}$$

For $R = 1$, standard Sobolev embeddings yield the result. And for $R > 1$, after localizing, one only needs to apply Sobolev embeddings in (r, ω). The decay is then a consequence of converting the volume element $dr\,d\omega$ to $dx = r^2\,dr\,d\omega$.

4 Proof of Theorem 1.1

We shall solve (1.1) via iteration. We set $u_0 \equiv 0$, and for $k \geq 1$, let u_k solve

$$\begin{cases} \Box u_k^I = a_{JK}^{I,\alpha} u_{k-1}^J \partial_\alpha u_{k-1}^K + b_{JK}^{I,\alpha\beta} \partial_\alpha u_{k-1}^J \partial_\beta u_{k-1}^K + A_{JK}^{I,\alpha\beta} u_{k-1}^K \partial_\alpha \partial_\beta u_k^J \\ \qquad\quad + B_{JK}^{I,\alpha\beta\gamma} \partial_\gamma u_{k-1}^K \partial_\alpha \partial_\beta u_k^J, \\ u_k(0,\cdot) = f, \quad \partial_t u_k(0,\cdot) = g. \end{cases} \tag{4.1}$$

Note that the right side of the equation is $F^I(u_{k-1}, u'_{k-1}, u''_k)$. We will show that the sequence (u_k) converges. The limit of this sequence is the desired solution.

We shall work with $N = 60$, though this is far from sharp. In the r^p-weighted estimates, we use $p = 1$ throughout.

To show that the sequence is convergent, we first show a certain boundedness. Relying on that, we next demonstrate that the sequence is Cauchy, which due to completeness of the spaces we are working in, finishes the proof.

4.1 Boundedness

For any fixed $T \leq T_\varepsilon$, we set

$$\begin{aligned} M_k \ &= \|\partial Z^{\leq 60} u_k\|_{L^\infty L^2} + \|Z^{\leq 60} u_k\|_{LE^1} + \|\not\partial (Z^{\leq 60} u_k)\|_{L^2 L^2} \\ &\quad + \|r^{-1} Z^{\leq 60} u_k\|_{L^2 L^2} + \|\langle r\rangle^{\frac{1}{2}} \not\partial Z^{\leq 60} u_k\|_{L^\infty L^2} + \|r^{-\frac{1}{2}} Z^{\leq 60} u_k\|_{L^\infty L^2} \\ &\quad + \sup_U \|r^{-\frac{1}{2}} \langle t-r\rangle^{-\frac{1}{2}} (\partial_t + \partial_r)(r Z^{\leq 60} u_k)\|_{L^2 L^2(X_U)} \\ &\quad + \sup_U \|r^{-\frac{1}{2}} (\log\langle r\rangle)^{-1} \langle t-r\rangle^{-\frac{1}{2}} Z^{\leq 60} u_k\|_{L^2 L^2(X_U)} \\ &\quad + \Big(\sum_{\tau \leq T} \sum_{R \leq \tau/2} \|\partial Z^{\leq 50} u_k\|^2_{L^2 L^2(\tilde{C}_\tau^R)}\Big)^{\frac{1}{2}} + \Big[\sum_{\tau \leq T} \sum_{R \leq \tau/2} \Big(R \|\not\partial \partial Z^{\leq 40} u_k\|_{L^2 L^2(\tilde{C}_\tau^R)}\Big)^2\Big]^{\frac{1}{2}} \\ &\quad + \sup_U \Big[\sum_{\tau \geq 4U} \Big(\frac{U^{\frac{1}{2}}}{\tau^{\frac{1}{2}} \log\langle\tau\rangle} \|\partial Z^{\leq 50} u_k\|_{L^2 L^2(\tilde{C}_\tau^U)}\Big)^2\Big]^{\frac{1}{2}} \\ &\quad + \sup_U \Big[\sum_{\tau \geq 4U} \Big(\frac{U^{\frac{1}{2}} \tau^{\frac{1}{2}}}{\log\langle\tau\rangle} \|\not\partial \partial Z^{\leq 40} u_k\|_{L^2 L^2(\tilde{C}_\tau^U)}\Big)^2\Big]^{\frac{1}{2}}. \end{aligned} \tag{4.2}$$

We call these terms I_k, II_k, …, XI_k, XII_k ,respectively. We shall argue inductively to show that

$$M_k \leq 2C_0\varepsilon \tag{4.3}$$

for a uniform constant C_0 provided that $T \leq T_\varepsilon$. Indeed, for a universal constant C_0, we shall show that

$$M_k^2 \leq (C_0\varepsilon)^2 + C(\log\langle T\rangle)^3 M_{k-1}^2 M_k + C(\log\langle T\rangle)^3 M_{k-1} M_k^2 + C(\log\langle T\rangle)^2 M_{k-1}^4 + C(\log\langle T\rangle)^2 M_{k-1}^2 M_k^2 + C(\log\langle T\rangle)^5 M_{k-1}^4 + C(\log\langle T\rangle)^5 M_{k-1}^3 M_k. \quad (4.4)$$

From this, it follows that $M_1 \leq C_0\varepsilon$. Then by the inductive hypothesis and (1.6), provided that c and ε are sufficiently small (compared to C_0), we obtain (4.3).

We briefly summarize the proof of (4.4) that is to follow. Note that terms I_k and II_k are bounded using the energy and integrated local energy estimate (2.2), while terms $III_k, \ldots, VIII_k$ represent the left side of (2.9). These eight terms are the principal portions. To prove (4.4), upon applying (2.2) and (2.9), the product rule will guarantee that one factor from each nonlinear term will be lower order (in terms of the number of vector fields). As this factor can afford additional vector fields, we may apply our decay estimates (3.1), (3.2), (3.3), or (3.7) to it.

Closing the argument requires that we obtain additional decay from the derivative that must be present on at least one factor of every nonlinear term. When this derivative is a "good" derivative $\not\partial$, this is relatively simple as the r^p-weight allows it to be bounded with a larger weight, which effectively provides additional decay to be used for the other factors. For the $\partial_t - \partial_r$ directions, provided that the factor can admit an additional vector field (3.5) yields additional decay. Here the decay is in $t - r$, and the use of the ghost weight allows our estimates to take advantage of this. Terms $IX_k, \ldots, XII_k$ of (4.2) are commonly occurring factors where such a procedure is utilized.

The resulting worst nonlinear term is when $u_{k-1}(\partial_t - \partial_r)Z^{\leq 60}u_{k-1}$ occurs within the right side of (2.9). Here one integrates $(\partial_t - \partial_r)$ by parts. When it lands on the lower order factor, the procedure based in (3.5) described above can be used. When it instead lands on the multiplier term, up to better behaved terms, $\Box u_k$ is reproduced. This term is replaced using the nonlinear equation, and quartic interactions result. The majority of the terms can be handled as above and the worst case is again the $u_{k-1}(\partial_t - \partial_r)Z^{\leq 60}u_{k-1}$ interactions. At this point, however, the two high-order factors can be combined using the chain rule $w\partial w = \frac{1}{2}\partial w^2$, and integration by parts can be used to move the derivative to the lower order factors where (3.5) can once again be applied.

We note that the extra logarithmic factor in term $VIII_k$ is largely responsible for our lifespan being $\exp(c/\varepsilon^{\frac{1}{3}})$ rather than $\exp(c/\varepsilon)$ as is known for scalar equations.

In our applications of (2.2) and (2.9), we set

$$h_J^{I,\alpha\beta} = A_{JK}^{I,\alpha\beta} u_{k-1}^K + B_{JK}^{I,\alpha\beta\gamma}\partial_\gamma u_{k-1}^K. \quad (4.5)$$

We proceed with establishing the necessary bound for each of $I_k, \ldots, XII_k$.

[$\mathbf{I_k} + \mathbf{II_k}$] : We begin by showing

$$I_k^2 + II_k^2 \leq (C_0\varepsilon)^2 + (\log\langle T\rangle)^{\frac{5}{2}} M_{k-1} M_k^2 + (\log\langle T\rangle)^{\frac{5}{2}} M_{k-1}^2 M_k. \quad (4.6)$$

Using (2.2) and (1.5), we have that

$$I_k^2 + II_k^2 \le C_0^2\varepsilon^2 + C\int_0^T\int |\Box_h Z^{\le 60}u_k|\Big(|\partial Z^{\le 60}u_k| + \frac{|Z^{\le 60}u_k|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+C\int_0^T\int \Big(|\partial\partial^{\le 1}u_{k-1}| + \frac{|\partial^{\le 1}u_{k-1}|}{\langle r\rangle}\Big)|\partial Z^{\le 60}u_k|\Big(|\partial Z^{\le 60}u_k| + \frac{|Z^{\le 60}u_k|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t. \tag{4.7}$$

We first note that we may apply (3.7) and the finite speed of propagation to see that the last term in the right of (4.7) is bounded by

$$\sum_{0\le j\lesssim \log\langle T\rangle} 2^{-j}\|r^{-1}(r\partial)^{\le 1}Z^{\le 3}u_{k-1}\|_{L^\infty L^2}\int_0^T\int_{A_{2j}} |\partial Z^{\le 60}u_k|\Big(|\partial Z^{\le 60}u_k| + \frac{|Z^{\le 60}u_k|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t.$$

The Schwarz inequality and a Hardy inequality then give that

$$\begin{aligned}&\int_0^T\int \Big(|\partial\partial^{\le 1}u_{k-1}| + \frac{|\partial^{\le 1}u_{k-1}|}{\langle r\rangle}\Big)|\partial Z^{\le 60}u_k|\Big(|\partial Z^{\le 60}u_k| + \frac{|Z^{\le 60}u_k|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &\lesssim \|\partial Z^{\le 3}u_{k-1}\|_{L^\infty L^2}\Big(\log\langle T\rangle\|Z^{\le 60}u_k\|_{LE^1}^2\Big)\\ &\lesssim I_{k-1}\Big(\log\langle T\rangle II_k^2\Big),\end{aligned} \tag{4.8}$$

which is controlled by the right side of (4.6).

To address the second term in the right side of (4.7), we note that

$$\begin{aligned}|\Box_h Z^{\le 60}u_k| \lesssim &\Big(|\partial Z^{\le 30}u_{k-1}| + |\partial Z^{\le 31}u_k|\Big)|\partial^{\le 1}Z^{\le 60}u_{k-1}|\\ &+|Z^{\le 31}u_{k-1}|\Big(|\partial Z^{\le 60}u_{k-1}| + |\partial Z^{\le 60}u_k|\Big).\end{aligned} \tag{4.9}$$

We first write

$$\begin{aligned}&\int_0^T\int |\Box_h Z^{\le 60}u_k|\Big(|\partial Z^{\le 60}u_k| + \frac{|Z^{\le 60}u_k|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &\lesssim \sum_{\tau\le T}\Big(\sum_{R\le\tau/2}\int\int_{C_\tau^R} |\Box_h Z^{\le 60}u_k|\Big(|\partial Z^{\le 60}u_k| + \frac{|Z^{\le 60}u_k|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \sum_{U\le\tau/4}\int\int_{C_\tau^U} |\Box_h Z^{\le 60}u_k|\Big(|\partial Z^{\le 60}u_k| + \frac{|Z^{\le 60}u_k|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t\Big).\end{aligned} \tag{4.10}$$

By (3.1) and (3.3), we have

$$\begin{aligned}
&\int\int_{C_\tau^R}\Big(|\partial Z^{\le 30}u_{k-1}|+|\partial Z^{\le 31}u_k|\Big)|\partial^{\le 1}Z^{\le 60}u_{k-1}|\Big(|\partial Z^{\le 60}u_k|+\frac{|Z^{\le 60}u_k|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim \Big(\|\partial Z^{\le 34}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}+\|\partial Z^{\le 35}u_k\|_{L^2L^2(\tilde{C}_\tau^R)}\\
&\quad +R\|\not\partial\partial Z^{\le 33}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}+R\|\not\partial\partial Z^{\le 34}u_k\|_{L^2L^2(\tilde{C}_\tau^R)}\Big)\\
&\quad \times\|\langle r\rangle^{-1}\partial^{\le 1}Z^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^R)}\|\langle r\rangle^{-1}(\partial Z^{\le 60}u_k, r^{-1}Z^{\le 60}u_k)\|_{L^2L^2(C_\tau^R)}.
\end{aligned}$$

Noting, for example, that

$$\|\langle r\rangle^{-1}\partial Z^{\le 60}u\|_{L^2L^2}^2=\sum_j 2^{-j}\Big(2^{-j}\|\partial Z^{\le 60}u\|_{L^2L^2([0,T]\times A_{2^j})}^2\Big)\lesssim\|Z^{\le 60}u\|_{LE^1}^2,$$

we thus have

$$\begin{aligned}
&\sum_{\tau\le T}\sum_{R\le\tau/2}\int\int_{C_\tau^R}\Big(|\partial Z^{\le 30}u_{k-1}|+|\partial Z^{\le 31}u_k|\Big)|\partial^{\le 1}Z^{\le 60}u_{k-1}|\Big(|\partial Z^{\le 60}u_k|+\frac{|Z^{\le 60}u_k|}{r}\Big)\,dx\,dt\\
&\lesssim \Big(IX_{k-1}+IX_k+X_{k-1}+X_k\Big)\Big(IV_{k-1}+II_{k-1}\Big)II_k. \qquad (4.11)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int\int_{C_\tau^R}|Z^{\le 31}u_{k-1}|\Big(|\partial Z^{\le 60}u_{k-1}|+|\partial Z^{\le 60}u_k|\Big)\Big(|\partial Z^{\le 60}u_k|+\frac{|Z^{\le 60}u_k|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim \Big(\|r^{-1}Z^{\le 35}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}+\|\not\partial Z^{\le 34}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big)\\
&\quad\times\|\langle r\rangle^{-\frac12}(\partial Z^{\le 60}u_k, r^{-1}Z^{\le 60}u_k)\|_{L^2L^2(C_\tau^R)}\\
&\quad\times\Big(\|\langle r\rangle^{-\frac12}\partial Z^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^R)}+\|\langle r\rangle^{-\frac12}\partial Z^{\le 60}u_k\|_{L^2L^2(C_\tau^R)}\Big).
\end{aligned}$$

Since the initial data are supported in $\{|x|\le 2\}$, we get

$$\begin{aligned}
\|\langle r\rangle^{-\frac12}\partial Z^{\le 60}u\|_{L^2L^2}^2 &\le \sum_{j\lesssim\log\langle T\rangle}2^{-j}\|\partial Z^{\le 60}u\|_{L^2L^2([0,T]\times A_{2^j})}^2\\
&\lesssim \log\langle T\rangle\|Z^{\le 60}u\|_{LE^1}^2. \qquad (4.12)
\end{aligned}$$

It follows that

$$\sum_{\tau\le T}\sum_{R\le\tau/2}\int\!\!\int_{C_\tau^R}|Z^{\le 31}u_{k-1}|\Big(|\partial Z^{\le 60}u_{k-1}|+|\partial Z^{\le 60}u_k|\Big)\Big(|\partial Z^{\le 60}u_k|+\frac{|Z^{\le 60}u_k|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$\lesssim \Big(IV_{k-1}+III_{k-1}\Big)\cdot II_k\cdot(\log\langle T\rangle)^{\frac12}\Big(II_{k-1}+II_k\Big). \tag{4.13}$$

By (3.2),

$$\int\!\!\int_{C_\tau^U}\Big(|\partial Z^{\le 30}u_{k-1}|+|\partial Z^{\le 31}u_k|\Big)|\partial^{\le 1}Z^{\le 60}u_{k-1}|\Big(|\partial Z^{\le 60}u_k|+\frac{|Z^{\le 60}u_k|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$\lesssim \frac{U^{\frac12}}{\tau^{\frac12}}\Big(\|\partial Z^{\le 34}u_{k-1}\|_{L^2L^2(\tilde C_\tau^U)}+\|\partial Z^{\le 35}u_k\|_{L^2L^2(\tilde C_\tau^U)}$$
$$+\tau\|\not\partial\partial Z^{\le 33}u_{k-1}\|_{L^2L^2(\tilde C_\tau^U)}+\tau\|\not\partial\partial Z^{\le 34}u_k\|_{L^2L^2(\tilde C_\tau^U)}\Big)$$
$$\times U^{-1}\|\langle r\rangle^{-\frac12}\partial^{\le 1}Z^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}\|\langle r\rangle^{-\frac12}(\partial Z^{\le 60}u_k,r^{-1}Z^{\le 60}u_k)\|_{L^2L^2(C_\tau^U)}.$$

Thus, using (4.12),

$$\sum_{\tau\le T}\sum_{U\le\tau/4}\int\!\!\int_{C_\tau^U}\Big(|\partial Z^{\le 30}u_{k-1}|+|\partial Z^{\le 31}u_k|\Big)$$
$$\times|\partial^{\le 1}Z^{\le 60}u_{k-1}|\Big(|\partial Z^{\le 60}u_k|+\frac{|Z^{\le 60}u_k|}{r}\Big)\,dx\,dt$$
$$\lesssim \log\langle T\rangle\Big(XI_{k-1}+XI_k+XII_{k-1}+XII_k\Big)\Big(\log\langle T\rangle VIII_{k-1}+(\log\langle T\rangle)^{\frac12}II_{k-1}\Big)$$
$$\times\Big((\log\langle T\rangle)^{\frac12}II_k\Big). \tag{4.14}$$

Relying on (3.2) again, we have

$$\int\!\!\int_{C_\tau^U}|Z^{\le 31}u_{k-1}|\Big(|\partial Z^{\le 60}u_{k-1}|+|\partial Z^{\le 60}u_k|\Big)\Big(|\partial Z^{\le 60}u_k|+\frac{|Z^{\le 60}u_k|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$\lesssim \frac{1}{U^{\frac12}\tau^{\frac12}}\Big(\|Z^{\le 35}u_{k-1}\|_{L^2L^2(\tilde C_\tau^U)}+\|(\partial_t+\partial_r)(rZ^{\le 34}u_{k-1})\|_{L^2L^2(\tilde C_\tau^U)}\Big)$$
$$\times\Big(\|\langle r\rangle^{-\frac12}\partial Z^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}+\|\langle r\rangle^{-\frac12}\partial Z^{\le 60}u_k\|_{L^2L^2(C_\tau^U)}\Big)$$
$$\times\|\langle r\rangle^{-\frac12}(\partial Z^{\le 60}u_k,r^{-1}Z^{\le 60}u_k)\|_{L^2L^2(C_\tau^U)}.$$

Upon using (4.12), it follows that

$$\sum_{\tau\le T}\sum_{U\le\tau/4}\int\!\!\int_{C_\tau^U}|Z^{\le 31}u_{k-1}|\Big(|\partial Z^{\le 60}u_{k-1}|+|\partial Z^{\le 60}u_k|\Big)\Big(|\partial Z^{\le 60}u_k|+\frac{|Z^{\le 60}u_k|}{|x|}\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$\lesssim \Big(\log\langle T\rangle VIII_{k-1}+VII_{k-1}\Big)\Big((\log\langle T\rangle)^{\frac12}II_{k-1}+(\log\langle T\rangle)^{\frac12}II_k\Big)\Big((\log\langle T\rangle)^{\frac12}II_k\Big). \tag{4.15}$$

By plugging bounds (4.8), (4.11), (4.13), (4.14), (4.15) into (4.7) and(4.10) we obtain the desired bound (4.6).

[$\mathbf{III_k}+\mathbf{IV_k}+\mathbf{V_k}+\mathbf{VI_k}+\mathbf{VII_k}+\mathbf{VIII_k}$]: Here, relying on (2.9), we show that

$$\begin{aligned}III_k^2+IV_k^2+V_k^2+VI_k^2+VII_k^2+VIII_k^2&\le (C_0\varepsilon)^2\\&+C(\log\langle T\rangle)^3M_{k-1}M_k^2+C(\log\langle T\rangle)^3M_{k-1}^2M_k\\&+C(\log\langle T\rangle)^5M_{k-1}^4+C(\log\langle T\rangle)^5M_{k-1}^3M_k.\end{aligned}\tag{4.16}$$

The first term in the right of (2.9) is bounded by $C_0^2\varepsilon^2$ due to (1.5). We will proceed, in order, to showing that each of the terms, other than the $\Box_h Z^{\le 60}u_k$ term, in the right side of (2.9) are bounded by

$$C\log\langle T\rangle^3\Big(M_{k-1}^2M_k+M_{k-1}M_k^2\Big). \tag{4.17}$$

We will argue separately that the $\Box_h Z^{\le 60}u_k$ term is bounded by the right side of (4.16), which will establish the desired bound.

To control the second term in the right side of (2.9), we will consider the integral at an arbitrary $t\in[0,T]$, and we fix a dyadic value τ so that $t\in[\tau,2\tau]$. For $1\le R\le\tau/2$, we can apply (3.1) and (3.3) and a Hardy inequality (after expanding the range of integration of the norm of $r^{-1}|Z^{\le 60}u_k|$ from A_R to $\mathbb{R}^3$) to see

$$\begin{aligned}&\int_{A_R} r|\partial^{\le 1}u_{k-1}||\partial Z^{\le 60}u_k|\Big(|\partial Z^{\le 60}u_k|+\frac{|Z^{\le 60}u_k|}{r}\Big)\,\mathrm{d}x\\&\lesssim\Big(R^{-1}\|Z^{\le 5}u_{k-1}\|_{L^2L^2(\tilde C_\tau^R)}+\|\partial Z^{\le 4}u_{k-1}\|_{L^2L^2(\tilde C_\tau^R)}\Big)\\&\qquad\|\partial Z^{\le 60}u_k(t,\cdot)\|_{L^2(A_R)}\|\partial Z^{\le 60}u_k\|_{L^\infty L^2}.\end{aligned}$$

And hence, using the Schwarz inequality,

$$\begin{aligned}&\sup_{t\in[0,T]}\sum_{R\le\tau/2}\int_{A_R} r|\partial^{\le 1}u_{k-1}||\partial Z^{\le 60}u_k|\Big(|\partial Z^{\le 60}u_k|+\frac{|Z^{\le 60}u_k|}{r}\Big)\,\mathrm{d}x\\&\lesssim\Big(IV_{k-1}+III_{k-1}\Big)I_k^2,\end{aligned}$$

which is dominated by (4.17). For the remainder of this term, (3.2) and a Hardy inequality show that

$$\begin{aligned}
&\sum_{1\le U\le \tau/4}\int_{\langle t-r\rangle\approx U} r|\partial^{\le 1}u_{k-1}||\partial Z^{\le 60}u_k|\Big(|\partial Z^{\le 60}u_k|+\frac{|Z^{\le 60}u_k|}{r}\Big)\,\mathrm{d}x\\
&\lesssim \sup_U\Big(\frac{1}{\tau^{\frac12}U^{\frac12}}\|Z^{\le 4}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}+\frac{1}{\tau^{\frac12}U^{\frac12}}\|(\partial_t+\partial_r)(rZ^{\le 3}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)\\
&\quad\times\|\partial Z^{\le 60}u_k(t,\,\cdot\,)\|_{L^2}^2.
\end{aligned}$$

The supremum of this is bounded by

$$\Big(\log\langle T\rangle VIII_{k-1}+VII_{k-1}\Big)I_k^2.$$

Thus,

$$\begin{aligned}
&\sup_{t\in[0,T]}\int r|\partial^{\le 1}u_{k-1}||\partial Z^{\le 60}u_k|\Big(|\partial Z^{\le 60}u_k|+\frac{|Z^{\le 60}u_k|}{r}\Big)\,\mathrm{d}x\\
&\lesssim \Big(IV_{k-1}+III_{k-1}+\log\langle T\rangle VIII_{k-1}+VII_{k-1}\Big)I_k^2.
\end{aligned}\tag{4.18}$$

We proceed to showing that

$$\begin{aligned}
&\sup_{U\ge 1}\sup_{t\in[0,T]}\Big|\int_0^t\int e^{-\sigma_U(t-r)}\Box_h Z^{\le 60}u_k\cdot\Big(\partial_t+\partial_r\Big)(rZ^{\le 60}u_k)\,\mathrm{d}x\,\mathrm{d}t\Big|\\
&\lesssim(\log\langle T\rangle)^3\Big(M_{k-1}^2M_k+M_{k-1}M_k^2\Big)\\
&\quad+(\log\langle T\rangle)^5M_{k-1}^4+(\log\langle T\rangle)^5M_{k-1}^3M_k.
\end{aligned}\tag{4.19}$$

Proof of (4.19). The most delicate terms in this analysis are those of the form $u_{k-1}(\partial_t-\partial_r)Z^{\le 60}u_{k-1}$. Here we have a bad derivative occurring at the highest regularity, and thus there is not room to apply, for example, (3.5) directly in order to get additional decay.

We begin by examining the other terms. To this end, we set $\omega=(1,-x/r)$ and note that

$$\begin{aligned}
&|\Box_h Z^{\le 60}u_k^I-a_{JK}^{I,\alpha}\omega_\alpha u_{k-1}^J(\partial_t-\partial_r)Z^{\le 60}u_{k-1}^K|\\
&\lesssim|Z^{\le 30}u_{k-1}||\not\partial Z^{\le 60}u_{k-1}|+|\partial Z^{\le 30}u_{k-1}||\partial^{\le 1}Z^{\le 60}u_{k-1}|\\
&\quad+|Z^{\le 30}u_{k-1}||\partial Z^{\le 59}u_{k-1}|+|\partial Z^{\le 31}u_k||\partial^{\le 1}Z^{\le 60}u_{k-1}|\\
&\quad+|\partial^{\le 1}Z^{\le 30}u_{k-1}||\partial^2Z^{\le 59}u_k|.
\end{aligned}\tag{4.20}$$

Using (3.1), (3.2), and (3.3) gives us that

$$\begin{aligned}
&\int\int_{C_\tau^R} r|Z^{\leq 30}u_{k-1}||\not\partial Z^{\leq 60}u_{k-1}|\Big(|\not\partial Z^{\leq 60}u_k| + r^{-1}|Z^{\leq 60}u_k|\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim \Big(R^{-1}\|Z^{\leq 34}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + \|\not\partial Z^{\leq 33}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big)\|\not\partial Z^{\leq 60}u_{k-1}\|_{L^2L^2(C_\tau^R)}\\
&\quad\times\Big(\|\not\partial Z^{\leq 60}u_k\|_{L^2L^2(C_\tau^R)} + \|r^{-1}Z^{\leq 60}u_k\|_{L^2L^2(C_\tau^R)}\Big),
\end{aligned}$$

and, respectively,

$$\begin{aligned}
&\int\int_{C_\tau^U} r|Z^{\leq 30}u_{k-1}||\not\partial Z^{\leq 60}u_{k-1}|\Big(|\not\partial Z^{\leq 60}u_k| + r^{-1}|Z^{\leq 60}u_k|\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim \Big(\frac{1}{U^{\frac12}\tau^{\frac12}}\|Z^{\leq 34}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \frac{1}{U^{\frac12}\tau^{\frac12}}\|(\partial_t+\partial_r)(rZ^{\leq 33}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)\\
&\quad\times\|\not\partial Z^{\leq 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}\\
&\quad\times\Big(\|\not\partial Z^{\leq 60}u_k\|_{L^2L^2(C_\tau^U)} + \|r^{-1}Z^{\leq 60}u_k\|_{L^2L^2(C_\tau^U)}\Big).
\end{aligned}$$

Upon summing over $R \leq \tau/2$, $U \leq \tau/4$ and $\tau \leq T$, we get

$$\begin{aligned}
&\int_0^T\int r|Z^{\leq 30}u_{k-1}||\not\partial Z^{\leq 60}u_{k-1}|\Big(|\not\partial Z^{\leq 60}u_k| + r^{-1}|Z^{\leq 60}u_k|\Big)\,dx\,dt\\
&\lesssim (IV_{k-1} + III_{k-1} + \log(T)VIII_{k-1} + VII_{k-1})III_{k-1}(III_k + IV_k).
\end{aligned}\tag{4.21}$$

Another application of (3.1) and (3.3) gives

$$\begin{aligned}
&\int\int_{C_\tau^R} r|\partial Z^{\leq 30}u_{k-1}||\partial^{\leq 1}Z^{\leq 60}u_{k-1}|\Big(|\not\partial Z^{\leq 60}u_k| + r^{-1}|Z^{\leq 60}u_k|\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim \Big(\|\partial Z^{\leq 34}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + R\|\not\partial\partial Z^{\leq 33}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big)\\
&\quad R^{-1}\|\partial^{\leq 1}Z^{\leq 60}u_{k-1}\|_{L^2L^2(C_\tau^R)}\\
&\quad\times\Big(\|\not\partial Z^{\leq 60}u_k\|_{L^2L^2(C_\tau^R)} + \|r^{-1}Z^{\leq 60}u_k\|_{L^2L^2(C_\tau^R)}\Big).
\end{aligned}$$

Similarly, using (3.2), we get

$$\begin{aligned}
&\int\int_{C_\tau^U} |\partial Z^{\leq 30}u_{k-1}||\partial^{\leq 1}Z^{\leq 60}u_{k-1}|\Big(|(\partial_t+\partial_r)(rZ^{\leq 60}u_k)|\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim \Big(\frac{U^{\frac12}}{\tau^{\frac12}}\|\partial Z^{\leq 34}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + U^{\frac12}\tau^{\frac12}\|(\partial_t+\partial_r)\partial Z^{\leq 33}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)
\end{aligned}$$

$$\times \frac{1}{U^{\frac{1}{2}}\tau^{\frac{1}{2}}}\|\partial^{\leq 1}Z^{\leq 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}$$
$$\times \frac{1}{U^{\frac{1}{2}}\tau^{\frac{1}{2}}}\|(\partial_t+\partial_r)(rZ^{\leq 60}u_k)\|_{L^2L^2(C_\tau^U)}.$$

Upon summing, these give

$$\begin{aligned}
&\int_0^T\int |\partial Z^{\leq 30}u_{k-1}||\partial^{\leq 1}Z^{\leq 60}u_{k-1}||(\partial_t+\partial_r)(rZ^{\leq 60}u_k)|\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim (IX_{k-1}+X_{k-1})(II_{k-1}+IV_{k-1})(III_k+IV_k)\\
&\quad+(\log\langle T\rangle)^2\Big(XI_{k-1}+XII_{k-1}\Big)\Big((\log\langle T\rangle)^{\frac{1}{2}}II_{k-1}+\log\langle T\rangle VIII_{k-1}\Big)VII_k.
\end{aligned}\tag{4.22}$$

Following the same argument, we also obtain

$$\begin{aligned}
&\int_0^T\int |\partial Z^{\leq 31}u_k||\partial^{\leq 1}Z^{\leq 60}u_{k-1}||(\partial_t+\partial_r)(rZ^{\leq 60}u_k)|\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim (IX_k+X_k)(II_{k-1}+IV_{k-1})(III_k+IV_k)\\
&\quad+(\log\langle T\rangle)^2\Big(XI_k+XII_k\Big)\Big((\log\langle T\rangle)^{\frac{1}{2}}II_{k-1}+\log\langle T\rangle VIII_{k-1}\Big)VII_k.
\end{aligned}\tag{4.23}$$

We now apply both (3.5) and (3.1) (and (3.3)) to see that

$$\begin{aligned}
&\int\int_{C_\tau^R} |Z^{\leq 30}u_{k-1}||\partial Z^{\leq 59}u_{k-1}||(\partial_t+\partial_r)(rZ^{\leq 60}u_k)|\,dx\,dt\\
&\lesssim \Big(\frac{1}{R}\|Z^{\leq 34}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}+\|\not{\partial}Z^{\leq 33}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big)\\
&\quad\Big(\frac{1}{R}\|Z^{\leq 60}u_{k-1}\|_{L^2L^2(C_\tau^R)}+\|\not{\partial}Z^{\leq 59}u_{k-1}\|_{L^2L^2(C_\tau^R)}\Big)\\
&\quad\times\Big(\|\not{\partial}Z^{\leq 60}u_k\|_{L^2L^2(C_\tau^R)}+\frac{1}{R}\|Z^{\leq 60}u_k\|_{L^2L^2(C_\tau^R)}\Big),
\end{aligned}$$

while (3.5) and (3.2) give

$$\begin{aligned}
&\int\int_{C_\tau^U} |Z^{\leq 30}u_{k-1}||\partial Z^{\leq 59}u_{k-1}||(\partial_t+\partial_r)(rZ^{\leq 60}u_k)|\,dx\,dt\\
&\lesssim \frac{1}{U^{\frac{1}{2}}\tau^{\frac{1}{2}}}\Big(\|Z^{\leq 34}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}+\|(\partial_t+\partial_r)(rZ^{\leq 33}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)
\end{aligned}$$

$$\times \frac{1}{U^{\frac{1}{2}}\tau^{\frac{1}{2}}}\Big(\|Z^{\leq 60}u_{k-1}\|_{L^2L^2(C_\tau^U)} + \|(\partial_t+\partial_r)(rZ^{\leq 60}u_{k-1})\|_{L^2L^2(C_\tau^U)}\Big)$$
$$\times \frac{1}{U^{\frac{1}{2}}\tau^{\frac{1}{2}}}\|(\partial_t+\partial_r)(rZ^{\leq 60}u_k)\|_{L^2L^2(C_\tau^U)}.$$

Upon summation, this gives

$$\begin{aligned}\int_0^T \int |Z^{\leq 30}u_{k-1}||\partial Z^{\leq 59}u_{k-1}||(\partial_t+\partial_r)(rZ^{\leq 60}u_k)|\,\mathrm{d}x\,\mathrm{d}t \\ \lesssim (IV_{k-1}+III_{k-1})^2(III_k+IV_k) \\ +\log\langle T\rangle\Big(\log\langle T\rangle VIII_{k-1}+VII_{k-1}\Big)^2 VII_k. \end{aligned} \tag{4.24}$$

Very similar arguments give

$$\begin{aligned}\int\int_{C_\tau^R} |\partial^{\leq 1}Z^{\leq 30}u_{k-1}||\partial^2 Z^{\leq 59}u_k||(\partial_t+\partial_r)(rZ^{\leq 60}u_k)|\,\mathrm{d}x\,\mathrm{d}t \\ \lesssim \Big(\frac{1}{R}\|Z^{\leq 35}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + \|\not\partial Z^{\leq 34}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big) \\ \times\Big(\frac{1}{R}\|\partial Z^{\leq 60}u_k\|_{L^2L^2(C_\tau^R)} + \|\not\partial Z^{\leq 60}u_k\|_{L^2L^2(C_\tau^R)}\Big) \\ \times\Big(\|\not\partial Z^{\leq 60}u_k\|_{L^2L^2(C_\tau^R)} + \frac{1}{R}\|Z^{\leq 60}u_k\|_{L^2L^2(C_\tau^R)}\Big)\end{aligned}$$

and

$$\begin{aligned}\int\int_{C_\tau^U} |\partial^{\leq 1}Z^{\leq 30}u_{k-1}||\partial^2 Z^{\leq 59}u_k||(\partial_t+\partial_r)(rZ^{\leq 60}u_k)|\,\mathrm{d}x\,\mathrm{d}t \\ \lesssim \frac{1}{U^{\frac{1}{2}}\tau^{\frac{1}{2}}}\Big(\|Z^{\leq 35}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \|(\partial_t+\partial_r)(rZ^{\leq 34}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\Big) \\ \times\frac{1}{U^{\frac{1}{2}}\tau^{\frac{1}{2}}}\Big(\|\partial Z^{\leq 60}u_k\|_{L^2L^2(C_\tau^U)} + \|(\partial_t+\partial_r)(rZ^{\leq 60}u_k)\|_{L^2L^2(C_\tau^U)}\Big) \\ \times\frac{1}{U^{\frac{1}{2}}\tau^{\frac{1}{2}}}\|(\partial_t+\partial_r)(rZ^{\leq 60}u_k)\|_{L^2L^2(C_\tau^U)},\end{aligned}$$

which yield

$$\begin{aligned}\int_0^T \int |\partial^{\leq 1}Z^{\leq 30}u_{k-1}||\partial^2 Z^{\leq 59}u_k||(\partial_t+\partial_r)(rZ^{\leq 60}u_k)|\,\mathrm{d}x\,\mathrm{d}t \\ \lesssim (IV_{k-1}+III_{k-1})(II_k+III_k)(III_k+IV_k) \\ +\Big(\log\langle T\rangle VIII_{k-1}+VII_{k-1}\Big)\Big(II_k+\log\langle T\rangle VII_k\Big)VII_k. \end{aligned} \tag{4.25}$$

As the right sides of (4.21), (4.22), (4.23), (4.24), and (4.25) are bounded by (4.17), to complete the bound (4.19), we need only examine

$$\sup_{U\geq 1}\sup_{t\in[0,T]}\Big|\int_0^t\int e^{-\sigma_U(t-r)}a_{JK}^{I,\alpha}\omega_\alpha u_{k-1}^J(\partial_t-\partial_r)Z^{\leq 60}u_{k-1}^K(\partial_t+\partial_r)(rZ^{\leq 60}u_k^I)\,\mathrm{d}x\,\mathrm{d}t\Big|.$$

The argument that we shall use here is reminiscent of normal forms.

We first integrate by parts to see that

$$\begin{aligned}
&\int_0^t\int e^{-\sigma_U(t-r)}a_{JK}^{I,\alpha}\omega_\alpha u_{k-1}^J(\partial_t-\partial_r)Z^{\leq 60}u_{k-1}^K(\partial_t+\partial_r)(rZ^{\leq 60}u_k^I)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int e^{-\sigma_U(t-r)}a_{JK}^{I,\alpha}\omega_\alpha u_{k-1}^J Z^{\leq 60}u_{k-1}^K(\partial_t+\partial_r)(rZ^{\leq 60}u_k^I)\,\mathrm{d}x\Big|_0^t\\
&\qquad+2\int_0^t\int r^{-1}e^{-\sigma_U(t-r)}a_{JK}^{I,\alpha}\omega_\alpha u_{k-1}^J Z^{\leq 60}u_{k-1}^K(\partial_t+\partial_r)(rZ^{\leq 60}u_k^I)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+2\int_0^t\int \sigma_U'(t-r)e^{-\sigma_U(t-r)}a_{JK}^{I,\alpha}\omega_\alpha u_{k-1}^J Z^{\leq 60}u_{k-1}^K(\partial_t+\partial_r)(rZ^{\leq 60}u_k^I)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad-\int_0^t\int e^{-\sigma_U(t-r)}a_{JK}^{I,\alpha}\omega_\alpha(\partial_t-\partial_r)u_{k-1}^J Z^{\leq 60}u_{k-1}^K(\partial_t+\partial_r)(rZ^{\leq 60}u_k^I)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad-\int_0^t\int e^{-\sigma_U(t-r)}a_{JK}^{I,\alpha}\omega_\alpha u_{k-1}^J Z^{\leq 60}u_{k-1}^K(\partial_t^2-\partial_r^2)(rZ^{\leq 60}u_k^I)\,\mathrm{d}x\,\mathrm{d}t. \qquad (4.26)
\end{aligned}$$

We shall proceed through arguments that will bound each term in (4.26).

For t fixed and $\tau\approx t$, we may apply (3.1), (3.3), and the Schwarz inequality to see that

$$\begin{aligned}
&\sum_{R\leq\tau/2}\int_{A_R} r|u_{k-1}||Z^{\leq 60}u_{k-1}|\Big(|\not\partial Z^{\leq 60}u_k|+r^{-1}|Z^{\leq 60}u_k|\Big)\,\mathrm{d}x\\
&\quad\lesssim\Big(\|r^{-1}Z^{\leq 4}u_{k-1}\|_{L^2L^2}+\|\not\partial Z^{\leq 3}u_{k-1}\|_{L^2L^2}\Big)\|r^{-\frac12}Z^{\leq 60}u_{k-1}(t,\,\cdot\,)\|_{L^2}\\
&\qquad\times\Big(\|\langle r\rangle^{\frac12}\not\partial Z^{\leq 60}u_k(t,\,\cdot\,)\|_{L^2}+\|r^{-\frac12}Z^{\leq 60}u_k(t,\,\cdot\,)\|_{L^2}\Big).
\end{aligned}$$

Using (3.2), we instead get

$$\sum_{U\leq\tau/4}\int_{\langle t-r\rangle\approx U} r|u_{k-1}||Z^{\leq 60}u_{k-1}|\Big(|\not\partial Z^{\leq 60}u_k|+r^{-1}|Z^{\leq 60}u_k|\Big)\,dx$$

$$\lesssim \sup_U \Big[\frac{1}{U^{\frac{1}{2}} \tau^{\frac{1}{2}}} \Big(\| Z^{\leq 4} u_{k-1} \|_{L^2 L^2(\tilde{C}_\tau^U)} + \| (\partial_t + \partial_r)(r Z^{\leq 3} u_{k-1}) \|_{L^2 L^2(\tilde{C}_\tau^U)} \Big) \Big]$$
$$\times \| r^{-\frac{1}{2}} Z^{\leq 60} u_{k-1}(t, \cdot) \|_{L^2}$$
$$\times \Big(\| \langle r \rangle^{\frac{1}{2}} \not\partial Z^{\leq 60} u_k(t, \cdot) \|_{L^2} + \| r^{-\frac{1}{2}} Z^{\leq 60} u_k(t, \cdot) \|_{L^2} \Big).$$

As such,

$$\sup_U \sup_{t \in [0,T]} \Big| \int e^{-\sigma_U (t-r)} a_{JK}^{I,\alpha} \omega_\alpha u_{k-1}^J Z^{\leq 60} u_{k-1}^K (\partial_t + \partial_r)(r Z^{\leq 60} u_k^I) \, \mathrm{d}x \Big|_0^t \Big|$$
$$\lesssim \Big(IV_{k-1} + III_{k-1} + \log\langle T \rangle VIII_{k-1} + VII_{k-1} \Big) VI_{k-1} (V_k + VI_k). \tag{4.27}$$

For the second term in the right of (4.26), provided $R \leq \tau/2$, we may apply (3.1) or (3.3) to see that

$$\int \int_{C_\tau^R} |u_{k-1}| |Z^{\leq 60} u_{k-1}| \Big(|\not\partial Z^{\leq 60} u_k| + r^{-1} |Z^{\leq 60} u_k| \Big) \, \mathrm{d}x \, \mathrm{d}t$$
$$\lesssim \Big(\| r^{-1} Z^{\leq 4} u_{k-1} \|_{L^2 L^2(\tilde{C}_\tau^R)} + \| \not\partial Z^{\leq 3} u_{k-1} \|_{L^2 L^2(\tilde{C}_\tau^R)} \Big) \| r^{-1} Z^{\leq 60} u_{k-1} \|_{L^2 L^2(C_\tau^R)}$$
$$\times \Big(\| \not\partial Z^{\leq 60} u_k \|_{L^2 L^2(C_\tau^R)} + \| r^{-1} Z^{\leq 60} u_k \|_{L^2 L^2(C_\tau^R)} \Big), \tag{4.28}$$

and for $U \leq \tau/4$, (3.2) gives

$$\int \int_{C_\tau^U} r^{-1} |u_{k-1}| |Z^{\leq 60} u_{k-1}| |(\partial_t + \partial_r)(r Z^{\leq 60} u_k)| \, \mathrm{d}x \, \mathrm{d}t$$
$$\lesssim \Big(\frac{1}{\tau} \| Z^{\leq 4} u_{k-1} \|_{L^2 L^2(\tilde{C}_\tau^U)} + \frac{1}{\tau} \| (\partial_t + \partial_r)(r Z^{\leq 3} u_{k-1}) \|_{L^2 L^2(\tilde{C}_\tau^U)} \Big)$$
$$\times \frac{1}{\tau} \| Z^{\leq 60} u_{k-1} \|_{L^2 L^2(C_\tau^U)}$$
$$\times \frac{1}{U^{\frac{1}{2}} \tau^{\frac{1}{2}}} \| (\partial_t + \partial_r)(r Z^{\leq 60} u_k) \|_{L^2 L^2(C_\tau^U)}. \tag{4.29}$$

Upon summing, this results in

$$\int_0^T \int r^{-1} |u_{k-1}| |Z^{\leq 60} u_{k-1}| |(\partial_t + \partial_r)(r Z^{\leq 60} u_k)| \, \mathrm{d}x \, \mathrm{d}t$$
$$\lesssim (IV_{k-1} + III_{k-1}) IV_{k-1} (III_k + IV_k + VII_k), \tag{4.30}$$

which suffices for the bound in the supremum (in both t and U) of the second term in the right side of (4.26).

Similar to (4.29), we estimate

$$\begin{aligned}
&\int\int_{C_\tau^U} \frac{1}{U}|u_{k-1}||Z^{\le 60}u_{k-1}||(\partial_t+\partial_r)(rZ^{\le 60}u_k)|\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim \Big(\frac{1}{\tau^{\frac12}U^{\frac12}}\|Z^{\le 4}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}+\frac{1}{\tau^{\frac12}U^{\frac12}}\|(\partial_t+\partial_r)(rZ^{\le 3}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)\\
&\quad\times\frac{1}{\tau^{\frac12}U^{\frac12}}\|Z^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}\\
&\quad\times\frac{1}{U^{\frac12}\tau^{\frac12}}\|(\partial_t+\partial_r)(rZ^{\le 60}u_k)\|_{L^2L^2(C_\tau^U)}.
\end{aligned}$$

Since

$$\sigma_U'(t-r)\lesssim\frac{1}{\tau}\ \text{ on } C_\tau^R,\quad \sigma_U'(t-r)\lesssim\frac{1}{U}\ \text{ on } C_\tau^U,$$

we may combine this with (4.28) to see that

$$\begin{aligned}
&\int_0^T\int \sigma_U'(t-r)|u_{k-1}||Z^{\le 60}u_{k-1}||(\partial_t+\partial_r)(rZ^{\le 60}u_k)|\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim (IV_{k-1}+III_{k-1})IV_{k-1}(III_k+IV_k)\\
&\quad+(\log\langle T\rangle)^2\Big(\log\langle T\rangle VIII_{k-1}+VII_{k-1}\Big)VIII_{k-1}VII_k,
\end{aligned}\tag{4.31}$$

which provides the appropriate bound for the supremums of the third term in (4.26).

As (4.22) suffices to bound the fourth term in the right side of (4.26), it remains to consider

$$\begin{aligned}
&\int_0^t\int e^{-\sigma_U(t-r)}a_{JK}^{I,\alpha}\omega_\alpha u_{k-1}^J Z^{\le 60}u_{k-1}^K(\partial_t^2-\partial_r^2)(rZ^{\le 60}u_k^I)\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^t\int e^{-\sigma_U(t-r)}a_{JK}^{I,\alpha}\omega_\alpha u_{k-1}^J Z^{\le 60}u_{k-1}^K\not\nabla\cdot\not\nabla(rZ^{\le 60}u_k^I)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^t\int re^{-\sigma_U(t-r)}a_{JK}^{I,\alpha}\omega_\alpha u_{k-1}^J Z^{\le 60}u_{k-1}^K h_{\tilde{J}}^{I,\beta\gamma}\partial_\beta\partial_\gamma Z^{\le 60}u_k^{\tilde{J}}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^t\int re^{-\sigma_U(t-r)}a_{JK}^{I,\alpha}\omega_\alpha u_{k-1}^J Z^{\le 60}u_{k-1}^K\Box_h Z^{\le 60}u_k^I\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{4.32}$$

For the first term in the right, we integrate by parts and use (1.8) to see that

$$\begin{aligned}
&\sup_U \sup_{t\in[0,T]} \Big| \int_0^t \int e^{-\sigma_U(t-r)} a^{I,\alpha}_{JK} \omega_\alpha u^J_{k-1} Z^{\le 60} u^K_{k-1} \not\nabla \cdot \not\nabla (r Z^{\le 60} u^I_k) \,\mathrm{d}x \,\mathrm{d}t \Big| \\
&\lesssim \int_0^T \int |Z^{\le 1} u_{k-1}| |Z^{\le 60} u_{k-1}| |\not\nabla Z^{\le 60} u_k| \,\mathrm{d}x \,\mathrm{d}t \\
&+ \int_0^T \int r |u_{k-1}| |\not\nabla Z^{\le 60} u_{k-1}| |\not\nabla Z^{\le 60} u_k| \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned}$$

The preceding bound (4.21) shows that the latter term is controlled by (4.17). And (3.1) (and (3.3)) and (3.2), respectively, give

$$\begin{aligned}
&\int \int_{C^R_\tau} |Z^{\le 1} u_{k-1}| |Z^{\le 60} u_{k-1}| |\not\nabla Z^{\le 60} u_k| \,\mathrm{d}x \,\mathrm{d}t \\
&\lesssim \Big(\|r^{-1} Z^{\le 5} u_{k-1}\|_{L^2L^2(\tilde{C}^R_\tau)} + \|\not\partial Z^{\le 4} u_{k-1}\|_{L^2L^2(\tilde{C}^R_\tau)} \Big) \\
&\quad \times \|r^{-1} Z^{\le 60} u_{k-1}\|_{L^2L^2(C^R_\tau)} \|\not\partial Z^{\le 60} u_k\|_{L^2L^2(C^R_\tau)}
\end{aligned}$$

and

$$\begin{aligned}
&\int \int_{C^U_\tau} |Z^{\le 1} u_{k-1}| |Z^{\le 60} u_{k-1}| |\not\nabla Z^{\le 60} u_k| \,\mathrm{d}x \,\mathrm{d}t \\
&\lesssim \Big(\frac{1}{U^{\frac{1}{2}} \tau^{\frac{1}{2}}} \|Z^{\le 5} u_{k-1}\|_{L^2L^2(\tilde{C}^U_\tau)} + \frac{1}{U^{\frac{1}{2}} \tau^{\frac{1}{2}}} \|(\partial_t + \partial_r)(r Z^{\le 4} u_{k-1})\|_{L^2L^2(\tilde{C}^U_\tau)} \Big) \\
&\quad \times \|r^{-1} Z^{\le 60} u_{k-1}\|_{L^2L^2(C^U_\tau)} \|\not\partial Z^{\le 60} u_k\|_{L^2L^2(C^U_\tau)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_0^T \int |Z^{\le 1} u_{k-1}| |Z^{\le 60} u_{k-1}| |\not\nabla Z^{\le 60} u_k| \,\mathrm{d}x \,\mathrm{d}t \\
&\lesssim (IV_{k-1} + III_{k-1} + \log\langle T\rangle VIII_{k-1} + VII_{k-1}) IV_{k-1} III_k,
\end{aligned}$$

which shows that

$$\begin{aligned}
&\sup_U \sup_{t\in[0,T]} \Big| \int_0^t \int e^{-\sigma_U(t-r)} a^{I,\alpha}_{JK} \omega_\alpha u^J_{k-1} Z^{\le 60} u^K_{k-1} \not\nabla \cdot \not\nabla (r Z^{\le 60} u^I_k) \,\mathrm{d}x \,\mathrm{d}t \Big| \\
&\lesssim \log\langle T\rangle M^2_{k-1} M_k. \qquad (4.33)
\end{aligned}$$

Another integration by parts gives

$$\begin{aligned}
&\sup_U \sup_{t\in[0,T]} \Big| \int_0^t \int r e^{-\sigma_U(t-r)} a^{I,\alpha}_{JK} \omega_\alpha u^J_{k-1} Z^{\leq 60} u^K_{k-1} h^{I,\beta\gamma}_{\tilde{J}} \partial_\beta \partial_\gamma Z^{\leq 60} u^{\tilde{J}}_k \,\mathrm{d}x\,\mathrm{d}t \Big| \\
&\lesssim \sup_{t\in[0,T]} \int r |\partial^{\leq 1} u_{k-1}|^2 |Z^{\leq 60} u_{k-1}| |\partial Z^{\leq 60} u_k| \,dx \\
&\quad + \int_0^T \int |\partial^{\leq 1} u_{k-1}|^2 |Z^{\leq 60} u_{k-1}| |\partial Z^{\leq 60} u_k| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int \frac{r}{\langle t-r\rangle} |\partial^{\leq 1} u_{k-1}|^2 |Z^{\leq 60} u_{k-1}| |\partial Z^{\leq 60} u_k| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int r |\partial^{\leq 1} u_{k-1}| |\partial \partial^{\leq 1} u_{k-1}| |Z^{\leq 60} u_{k-1}| |\partial Z^{\leq 60} u_k| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int r |\partial^{\leq 1} u_{k-1}|^2 |\partial Z^{\leq 60} u_{k-1}| |\partial Z^{\leq 60} u_k| \,\mathrm{d}x\,\mathrm{d}t. \qquad (4.34)
\end{aligned}$$

If we argue precisely as in (4.27), we see that

$$\begin{aligned}
&\sup_{t\in[0,T]} \int r |\partial^{\leq 1} u_{k-1}|^2 |Z^{\leq 60} u_{k-1}| |\partial Z^{\leq 60} u_k| \,\mathrm{d}x \\
&\lesssim \Big(IV_{k-1} + III_{k-1} + \log\langle T\rangle VIII_{k-1} + VII_{k-1} \Big) \\
&\quad VI_{k-1} \| \langle r\rangle^{\frac{1}{2}} |\partial^{\leq 1} u_{k-1}| |\partial Z^{\leq 60} u_k| \|_{L^\infty L^2}.
\end{aligned}$$

Subsequently applying (3.7) gives that

$$\| \langle r\rangle^{\frac{1}{2}} |\partial^{\leq 1} u_{k-1}| |\partial Z^{\leq 60} u_k| \|_{L^\infty L^2} \lesssim \| r^{-\frac{1}{2}} Z^{\leq 3} u_{k-1} \|_{L^\infty L^2} \| \partial Z^{\leq 60} u_k \|_{L^\infty L^2}.$$

And hence,

$$\begin{aligned}
&\sup_{t\in[0,T]} \int r |\partial^{\leq 1} u_{k-1}|^2 |Z^{\leq 60} u_{k-1}| |\partial Z^{\leq 60} u_k| \,\mathrm{d}x \\
&\lesssim \Big(IV_{k-1} + III_{k-1} + \log\langle T\rangle VIII_{k-1} + VII_{k-1} \Big) VI_{k-1} VI_{k-1} I_k. \qquad (4.35)
\end{aligned}$$

For the remaining terms in (4.34), we continue to apply (3.1), (3.2), and (3.3) repeatedly. These give

$$\begin{aligned}
&\int\int_{C_\tau^R} |\partial^{\le 1}u_{k-1}|^2 |Z^{\le 60}u_{k-1}||\partial Z^{\le 60}u_k|\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim \Big(\|r^{-1}Z^{\le 5}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + \|\partial Z^{\le 4}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big)^2\\
&\quad\times \|r^{-1}Z^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^R)} \frac{1}{R^{\frac12}} \|r^{-\frac12}\partial Z^{\le 60}u_k\|_{L^2L^2(C_\tau^R)}
\end{aligned}$$

and

$$\begin{aligned}
&\int\int_{C_\tau^U} |\partial^{\le 1}u_{k-1}|^2 |Z^{\le 60}u_{k-1}||\partial Z^{\le 60}u_k|\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim \Big(\frac{1}{U^{\frac12}\tau^{\frac12}}\|Z^{\le 5}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \frac{1}{U^{\frac12}\tau^{\frac12}}\|(\partial_t+\partial_r)(rZ^{\le 4}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)^2\\
&\quad\times \|r^{-1}Z^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^U)} \frac{1}{\tau^{\frac12}} \|r^{-\frac12}\partial Z^{\le 60}u_k\|_{L^2L^2(C_\tau^U)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int\int_{C_\tau^U} \frac{r}{\langle t-r\rangle} |\partial^{\le 1}u_{k-1}|^2 |Z^{\le 60}u_{k-1}||\partial Z^{\le 60}u_k|\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim \Big(\frac{1}{U^{\frac12}\tau^{\frac12}}\|Z^{\le 5}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \frac{1}{U^{\frac12}\tau^{\frac12}}\|(\partial_t+\partial_r)(rZ^{\le 4}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)^2\\
&\quad\times U^{-1}\|r^{-\frac12}Z^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^U)} \|r^{-\frac12}\partial Z^{\le 60}u_k\|_{L^2L^2(C_\tau^U)}.
\end{aligned}$$

These combine to give

$$\begin{aligned}
&\int_0^T \int |\partial^{\le 1}u_{k-1}|^2 |Z^{\le 60}u_{k-1}||\partial Z^{\le 60}u_k|\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int\int_{C_\tau^U} \frac{r}{\langle t-r\rangle} |\partial^{\le 1}u_{k-1}|^2 |Z^{\le 60}u_{k-1}||\partial Z^{\le 60}u_k|\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim (IV_{k-1}+III_{k-1})^2 IV_{k-1} II_k + (\log\langle T\rangle VIII_{k-1} + VII_{k-1})^2\\
&\quad\times\Big(IV_{k-1}II_k + \log\langle T\rangle VIII_{k-1}II_k\Big).
\end{aligned} \tag{4.36}$$

By related arguments,

$$\begin{aligned}
&\int\int_{C_\tau^R} r|\partial^{\le 1}u_{k-1}||\partial\partial^{\le 1}u_{k-1}||Z^{\le 60}u_{k-1}||\partial Z^{\le 60}u_k|\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim \Big(\|r^{-1}Z^{\le 5}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}+\|\not\partial Z^{\le 4}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big)\|r^{-1}Z^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^R)}\\
&\quad\times\Big(\|\partial Z^{\le 5}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}+R\|\not\partial\partial Z^{\le 4}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big)\\
&\quad R^{-\frac12}\|\langle r\rangle^{-\frac12}\partial Z^{\le 60}u_k\|_{L^2L^2(C_\tau^R)}
\end{aligned}$$

and

$$\begin{aligned}
&\int\int_{C_\tau^U} r|\partial^{\le 1}u_{k-1}||\partial\partial^{\le 1}u_{k-1}||Z^{\le 60}u_{k-1}||\partial Z^{\le 60}u_k|\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim \frac{1}{U^{\frac12}}\Big(\frac{1}{U^{\frac12}\tau^{\frac12}}\|Z^{\le 5}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}+\frac{1}{U^{\frac12}\tau^{\frac12}}\|(\partial_t+\partial_r)(rZ^{\le 4}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)\\
&\quad\times\Big(\frac{U^{\frac12}}{\tau^{\frac12}}\|\partial Z^{\le 5}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}+U^{\frac12}\tau^{\frac12}\|\not\partial\partial Z^{\le 4}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)\\
&\quad\times\frac{1}{U^{\frac12}\tau^{\frac12}}\|Z^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}\frac{1}{\tau^{\frac12}}\|\partial Z^{\le 60}u_k\|_{L^2L^2(C_\tau^U)},
\end{aligned}$$

which gives

$$\begin{aligned}
&\int_0^T\int r|\partial^{\le 1}u_{k-1}||\partial\partial^{\le 1}u_{k-1}||Z^{\le 60}u_{k-1}||\partial Z^{\le 60}u_k|\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim (IV_{k-1}+III_{k-1})(IX_{k-1}+X_{k-1})IV_{k-1}II_k\\
&\quad+(\log\langle T\rangle)^2(\log\langle T\rangle VIII_{k-1}+VII_{k-1})\\
&\quad\times(XI_{k-1}+XII_{k-1})VIII_{k-1}II_k.
\end{aligned}\tag{4.37}$$

For the last term in (4.34), we get

$$\begin{aligned}
&\int\int_{C_\tau^R} r|\partial^{\le 1}u_{k-1}|^2|\partial Z^{\le 60}u_{k-1}||\partial Z^{\le 60}u_k|\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim \Big(\|r^{-1}Z^{\le 5}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}+\|\not\partial Z^{\le 4}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big)^2\\
&\quad\times\|r^{-\frac12}\partial Z^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^R)}\|r^{-\frac12}\partial Z^{\le 60}u_k\|_{L^2L^2(C_\tau^R)}
\end{aligned}$$

and

$$\begin{aligned}&\int\int_{C_\tau^U} r|\partial^{\leq 1}u_{k-1}|^2|\partial Z^{\leq 60}u_{k-1}||\partial Z^{\leq 60}u_k|\,\mathrm{d}x\,\mathrm{d}t\\&\lesssim \Big(\frac{1}{U^{\frac{1}{2}}\tau^{\frac{1}{2}}}\|Z^{\leq 5}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}+\frac{1}{U^{\frac{1}{2}}\tau^{\frac{1}{2}}}\|(\partial_t+\partial_r)(rZ^{\leq 4}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)^2\\&\quad\times\|r^{-\frac{1}{2}}\partial Z^{\leq 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}\|r^{-\frac{1}{2}}\partial Z^{\leq 60}u_k\|_{L^2L^2(C_\tau^U)},\end{aligned}$$

yielding

$$\begin{aligned}&\int_0^T\int r|\partial^{\leq 1}u_{k-1}|^2|\partial Z^{\leq 60}u_{k-1}||\partial Z^{\leq 60}u_k|\,dx\,dt\\&\lesssim (IV_{k-1}+III_{k-1})^2II_{k-1}II_k\\&\quad+\log\langle T\rangle(\log\langle T\rangle VIII_{k-1}+VII_{k-1})^2II_{k-1}II_k.\end{aligned}\tag{4.38}$$

The combination of (4.35), (4.36), (4.37), and (4.38) then establishes that

$$\begin{aligned}&\sup_U\sup_{t\in[0,T]}\Big|\int_0^t\int re^{-\sigma_U(t-r)}a_{JK}^{I,\alpha}\omega_\alpha u_{k-1}^J Z^{\leq 60}u_{k-1}^K h_{\tilde{J}}^{I,\alpha\beta}\partial_\alpha\partial_\beta Z^{\leq 60}u_k^{\tilde{J}}\,\mathrm{d}x\,\mathrm{d}t\Big|\\&\lesssim(\log\langle T\rangle)^3M_{k-1}^3M_k.\end{aligned}\tag{4.39}$$

We now turn our attention to

$$\sup_U\sup_{t\in[0,T]}\Big|\int_0^t\int re^{-\sigma_U(t-r)}a_{JK}^{I,\alpha}\omega_\alpha u_{k-1}^J Z^{\leq 60}u_{k-1}^K\Box_h Z^{\leq 60}u_k^I\,\mathrm{d}x\,\mathrm{d}t\Big|.$$

We shall first show

$$\begin{aligned}&\sup_U\sup_{t\in[0,T]}\Big|\int_0^t\int re^{-\sigma_U(t-r)}a_{JK}^{I,\alpha}\omega_\alpha u_{k-1}^J Z^{\leq 60}u_{k-1}^K\\&\Big(\Box_h Z^{\leq 60}u_k^I-a_{\tilde{J}\tilde{K}}^{I,\beta}\omega_\beta u_{k-1}^{\tilde{J}}(\partial_t-\partial_r)Z^{\leq 60}u_{k-1}^{\tilde{K}}\Big)\,\mathrm{d}x\,\mathrm{d}t\Big|\\&\lesssim(\log\langle T\rangle)^5M_{k-1}^4+(\log\langle T\rangle)^5M_{k-1}^3M_k.\end{aligned}\tag{4.40}$$

By (4.20), we have

$$\begin{aligned}&\sup_U\sup_{t\in[0,T]}\Big|\int_0^t\int re^{-\sigma_U(t-r)}a_{JK}^{I,\alpha}\omega_\alpha u_{k-1}^J Z^{\leq 60}u_{k-1}^K\\&\quad\times\Big(\Box_h Z^{\leq 60}u_k^I-a_{\tilde{J}\tilde{K}}^{I,\beta}\omega_\beta u_{k-1}^{\tilde{J}}(\partial_t-\partial_r)Z^{\leq 60}u_{k-1}^{\tilde{K}}\Big)\,\mathrm{d}x\,\mathrm{d}t\Big|\end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_0^T \int r|u_{k-1}||Z^{\leq 60}u_{k-1}||Z^{\leq 30}u_{k-1}||\not\partial Z^{\leq 60}u_{k-1}|\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int r|u_{k-1}||Z^{\leq 60}u_{k-1}||\partial Z^{\leq 30}u_{k-1}||\partial^{\leq 1}Z^{\leq 60}u_{k-1}|\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int r|u_{k-1}||Z^{\leq 60}u_{k-1}||\partial Z^{\leq 31}u_{k}||\partial^{\leq 1}Z^{\leq 60}u_{k-1}|\,\mathrm{d}x\,\mathrm{d}t \\
&\int_0^T \int r|u_{k-1}||Z^{\leq 60}u_{k-1}||\partial^{\leq 1}Z^{\leq 30}u_{k-1}||\partial^2 Z^{\leq 59}u_{k}|\,\mathrm{d}x\,\mathrm{d}t. \qquad (4.41)
\end{aligned}
$$

We note that (3.1) and (3.3) give

$$
\begin{aligned}
&\||u_{k-1}||Z^{\leq 60}u_{k-1}|\|_{L^2L^2(C_\tau^R)} \\
&\lesssim \Big(\|r^{-1}Z^{\leq 4}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + \|\not\partial Z^{\leq 3}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big)\|r^{-1}Z^{\leq 60}u_{k-1}\|_{L^2L^2(C_\tau^R)} \qquad (4.42)
\end{aligned}
$$

and (3.2) gives

$$
\begin{aligned}
&\frac{\tau^{\frac{1}{2}}}{U^{\frac{1}{2}}}\||u_{k-1}||Z^{\leq 60}u_{k-1}|\|_{L^2L^2(C_\tau^U)} \\
&\lesssim \frac{1}{\tau^{\frac{1}{2}}U^{\frac{1}{2}}}\Big(\|Z^{\leq 4}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \|(\partial_t+\partial_r)(rZ^{\leq 3}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\Big) \\
&\quad \times \frac{1}{U^{\frac{1}{2}}\tau^{\frac{1}{2}}}\|Z^{\leq 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}. \qquad (4.43)
\end{aligned}
$$

Arguing as in (4.21), (4.22), (4.23), and (4.25) where

$$
\|\not\partial Z^{\leq 60}u_k\|_{L^2L^2(C_\tau^R)}, \quad \frac{1}{U^{\frac{1}{2}}\tau^{\frac{1}{2}}}\|(\partial_t+\partial_r)(rZ^{\leq 60}u_k)\|_{L^2L^2(C_\tau^U)}
$$

are replaced by (4.42) and (4.43), respectively, we immediately get (4.40).

To complete the proof of (4.19), we now consider

$$
\begin{aligned}
&\int_0^t \int re^{-\sigma_U(t-r)}a_{JK}^{I,\alpha}\omega_\alpha u_{k-1}^J Z^{\leq 60}u_{k-1}^K\Big(a_{\tilde{J}\tilde{K}}^{I,\beta}\omega_\beta u_{k-1}^{\tilde{J}}(\partial_t-\partial_r)Z^{\leq 60}u_{k-1}^{\tilde{K}}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^t \int re^{-\sigma_U(t-r)}u_{k-1}^J u_{k-1}^{\tilde{J}}(\partial_t-\partial_r)\Big[a_{JK}^{I,\alpha}\omega_\alpha a_{\tilde{J}\tilde{K}}^{I,\beta}\omega_\beta Z^{\leq 60}u_{k-1}^K Z^{\leq 60}u_{k-1}^{\tilde{K}}\Big]\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Integration by parts shows that

$$\begin{aligned}
&\sup_U \sup_{t\in[0,T]} \Big| \int_0^t \int r e^{-\sigma_U (t-r)} a_{JK}^{I,\alpha} \omega_\alpha u_{k-1}^J Z^{\le 60} u_{k-1}^K \\
&\quad \Big(a_{\tilde J \tilde K}^{I,\beta} \omega_\beta u_{k-1}^{\tilde J} (\partial_t - \partial_r) Z^{\le 60} u_{k-1}^{\tilde K} \Big) \, \mathrm{d}x \, \mathrm{d}t \Big| \\
&\lesssim \sup_{t\in[0,T]} \int r |u_{k-1}|^2 |Z^{\le 60} u_{k-1}|^2 \, \mathrm{d}x + \int_0^T \int \frac{r}{\langle t-r\rangle} |u_{k-1}|^2 |Z^{\le 60} u_{k-1}|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^T \int |u_{k-1}|^2 |Z^{\le 60} u_{k-1}|^2 \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int r |u_{k-1}| |\partial u_{k-1}| |Z^{\le 60} u_{k-1}|^2 \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned} \tag{4.44}$$

For the first term, we first consider a fixed t and set $\tau \approx t$. Then by (3.1) (and (3.3)),

$$\begin{aligned}
&\int_{A_R} r |u_{k-1}|^2 |Z^{\le 60} u_{k-1}|^2 \, \mathrm{d}x \lesssim \Big(\| r^{-1} Z^{\le 4} u_{k-1} \|_{L^2 L^2(\tilde C_\tau^R)} + \| \partial Z^{\le 3} u_{k-1} \|_{L^2 L^2(\tilde C_\tau^R)} \Big)^2 \\
&\quad \| r^{-\frac12} Z^{\le 60} u_{k-1}(t,\cdot) \|_{L^2}^2,
\end{aligned}$$

and by (3.2)

$$\begin{aligned}
&\int_{\langle t-r\rangle \approx U} r |u_{k-1}|^2 |Z^{\le 60} u_{k-1}|^2 \, \mathrm{d}x \\
&\lesssim \Big(\frac{1}{U^{\frac12} \tau^{\frac12}} \| Z^{\le 4} u_{k-1} \|_{L^2 L^2(\tilde C_\tau^U)} + \frac{1}{U^{\frac12} \tau^{\frac12}} \| (\partial_t + \partial_r) r Z^{\le 3} u_{k-1} \|_{L^2 L^2(\tilde C_\tau^U)} \Big)^2 \\
&\quad \times \| r^{-\frac12} Z^{\le 60} u_{k-1}(t,\cdot) \|_{L^2(\{\langle t-r\rangle \approx U\})}^2.
\end{aligned}$$

Upon summing over $R \le \tau/2$, $U \le \tau/4$ and then taking a supremum in t, we obtain

$$\begin{aligned}
&\sup_{t\in[0,T]} \int r |u_{k-1}|^2 |Z^{\le 60} u_{k-1}|^2 \, \mathrm{d}x \lesssim (IV_{k-1} + III_{k-1})^2 V I_{k-1}^2 \\
&\quad + (\log\langle T\rangle V III_{k-1} + V II_{k-1})^2 V I_{k-1}^2.
\end{aligned} \tag{4.45}$$

For the second term in the right of (4.44), subsequently applying (3.1) (and (3.3)) and (3.2) when $R \le \tau/2$, $U \le \tau/4$ gives

$$\int\int_{C_\tau^R} \frac{r}{\langle t-r\rangle}|u_{k-1}|^2|Z^{\le 60}u_{k-1}|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\lesssim \Big(\|r^{-1}Z^{\le 4}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + \|\partial Z^{\le 3}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big)^2$$
$$\times \|r^{-1}Z^{\le 60}u_{k-1}\|^2_{L^2L^2(C_\tau^R)} \tag{4.46}$$

and

$$\int\int_{C_\tau^U} \frac{r}{\langle t-r\rangle}|u_{k-1}|^2|Z^{\le 60}u_{k-1}|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\lesssim \Big(\frac{1}{U^{\frac12}\tau^{\frac12}}\|Z^{\le 4}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \frac{1}{U^{\frac12}\tau^{\frac12}}\|(\partial_t+\partial_r)(rZ^{\le 3}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)^2$$
$$\times\Big(\frac{1}{U^{\frac12}\tau^{\frac12}}\|Z^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}\Big)^2.$$

Upon summing, we obtain

$$\int_0^T\int \frac{r}{\langle t-r\rangle}|u_{k-1}|^2|Z^{\le 60}u_{k-1}|^2\,\mathrm{d}x\,\mathrm{d}t \lesssim (IV_{k-1}+III_{k-1})^2IV_{k-1}^2$$
$$+(\log\langle T\rangle)^3\Big(\log\langle T\rangle VIII_{k-1}+VII_{k-1}\Big)^2VIII_{k-1}^2. \tag{4.47}$$

Moreover,

$$\int\int_{C_\tau^U} |u_{k-1}|^2|Z^{\le 60}u_{k-1}|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\lesssim \Big(\frac{1}{U^{\frac12}\tau^{\frac12}}\|Z^{\le 4}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \frac{1}{U^{\frac12}\tau^{\frac12}}\|(\partial_t+\partial_r)(rZ^{\le 3}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)^2$$
$$\times\|r^{-1}Z^{\le 60}u_{k-1}\|^2_{L^2L^2(C_\tau^U)}.$$

Hence when combined with (4.46), this gives

$$\int_0^T\int |u_{k-1}|^2|Z^{\le 60}u_{k-1}|^2\,\mathrm{d}x\,\mathrm{d}t \lesssim (IV_{k-1}+III_{k-1})^2IV_{k-1}^2+\Big(\log\langle T\rangle VIII_{k-1}$$
$$+VII_{k-1}\Big)^2IV_{k-1}^2. \tag{4.48}$$

We may use the estimate on the second term in the right side of (4.41) to bound the last term in (4.44). Combining this with (4.45), (4.47), and (4.48) then yields

$$\begin{aligned}\sup_U \sup_{t\in[0,T]} \Big| \int_0^t \int & re^{-\sigma_U(t-r)} a_{JK}^{I,\alpha} \omega_\alpha u_{k-1}^J Z^{\le 60} u_{k-1}^K \\ & \Big(a_{\tilde J \tilde K}^{I,\beta} \omega_\beta u_{k-1}^{\tilde J} (\partial_t - \partial_r) Z^{\le 60} u_{k-1}^{\tilde K}\Big)\, \mathrm{d}x\, \mathrm{d}t \Big| \\ & \lesssim (\log\langle T\rangle)^5 M_{k-1}^4,\end{aligned}$$

which completes the proof of (4.19).

To finish the proof of (4.16), we consider the remainder of the terms in the right side of (2.9) and show that they are each controlled by (4.17).

By (3.1) and (3.3),

$$\begin{aligned}\int\int_{C_\tau^R} & \Big(|\partial Z^{\le 1} u_{k-1}| + \frac{|Z^{\le 1} u_{k-1}|}{r}\Big) |\partial Z^{\le 60} u_k| |(\partial_t + \partial_r)(r Z^{\le 60} u_k)|\, \mathrm{d}x\, \mathrm{d}t \\ & \lesssim \Big(\|r^{-1} Z^{\le 5} u_{k-1}\|_{L^2L^2(\tilde C_\tau^R)} + \|\partial Z^{\le 5} u_{k-1}\|_{L^2L^2(\tilde C_\tau^R)} + R\|\not\partial \partial Z^{\le 4} u_{k-1}\|_{L^2L^2(\tilde C_\tau^R)}\Big) \\ & \quad \times \|r^{-1}\partial Z^{\le 60} u_k\|_{L^2L^2(C_\tau^R)} \Big(\|\not\partial Z^{\le 60} u_k\|_{L^2L^2(C_\tau^R)} + \|r^{-1} Z^{\le 60} u_k\|_{L^2L^2(C_\tau^R)}\Big).\end{aligned}$$

And by (3.2),

$$\begin{aligned}\int\int_{C_\tau^U} & \Big(|\partial Z^{\le 1} u_{k-1}| + \frac{|Z^{\le 1} u_{k-1}|}{r}\Big) |\partial Z^{\le 60} u_k| |(\partial_t + \partial_r)(r Z^{\le 60} u_k)|\, \mathrm{d}x\, \mathrm{d}t \\ & \lesssim \Big(\|r^{-1} Z^{\le 5} u_{k-1}\|_{L^2L^2(\tilde C_\tau^U)} + \frac{U^{\frac12}}{\tau^{\frac12}} \|\partial Z^{\le 5} u_{k-1}\|_{L^2L^2(\tilde C_\tau^U)} \\ & \quad + U^{\frac12}\tau^{\frac12} \|\not\partial \partial Z^{\le 4} u_{k-1}\|_{L^2L^2(\tilde C_\tau^U)}\Big) \\ & \quad \times \frac{1}{U^{\frac12}} \|\langle r\rangle^{-\frac12} \partial Z^{\le 60} u_k\|_{L^2L^2(C_\tau^U)} \frac{1}{U^{\frac12}\tau^{\frac12}} \|(\partial_t + \partial_r)(r Z^{\le 60} u_k)\|_{L^2L^2(C_\tau^U)}.\end{aligned}$$

Upon summing over $R \le \tau/2$, $U \le \tau/4$ and $\tau \le T$, we get

$$\begin{aligned}\int_0^T\int & \Big(|\partial Z^{\le 1} u_{k-1}| + \frac{|Z^{\le 1} u_{k-1}|}{r}\Big) |\partial Z^{\le 60} u_k| |(\partial_t + \partial_r)(r Z^{\le 60} u_k)|\, \mathrm{d}x\, \mathrm{d}t \\ & \lesssim (IV_{k-1} + IX_{k-1} + X_{k-1}) II_k (III_k + IV_k) \\ & \quad + \Big(IV_{k-1} + \log\langle T\rangle XI_{k-1} + \log\langle T\rangle XII_{k-1}\Big) II_k VII_k. \qquad (4.49)\end{aligned}$$

For the fifth term in the right side of (2.9), applying (3.7) and the Schwarz inequality, we have

$$\begin{aligned}
&\int_0^T \int |\partial^{\le 1} u_{k-1}||\partial Z^{\le 60} u_k|\Big(|\not\nabla Z^{\le 60} u_k| + \frac{|Z^{\le 60} u_k|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\lesssim \|\langle r\rangle^{-1} Z^{\le 3} u_{k-1}\|_{L^2L^2}\|\partial Z^{\le 60} u_k\|_{L^\infty L^2}\Big(\|\not\nabla Z^{\le 60} u_k\|_{L^2L^2} + \|r^{-1} Z^{\le 60} u_k\|_{L^2L^2}\Big) \\
&\lesssim IV_{k-1} I_k (III_k + IV_k). \qquad (4.50)
\end{aligned}$$

And for the sixth term, using (3.7),

$$\begin{aligned}
&\int_0^T \int \langle t-r\rangle^{-1} |\partial^{\le 1} u_{k-1}||\partial Z^{\le 60} u_k||(\partial_t + \partial_r)(r Z^{\le 60} u_k)|\,\mathrm{d}x\,\mathrm{d}t \\
&\lesssim \log\langle T\rangle \Big(\sup_U \|\langle r\rangle^{-\frac12}\langle t-r\rangle^{-\frac12} Z^{\le 3} u_{k-1}\|_{L^2L^2(X_U)}\Big)\|\partial Z^{\le 60} u_k\|_{L^\infty L^2} \qquad (4.51) \\
&\quad \times \sup_U \Big(\|\langle r\rangle^{-\frac12}\langle t-r\rangle^{-\frac12}(\partial_t + \partial_r)(r Z^{\le 60} u_k)\|_{L^2L^2(X_U)}\Big) \\
&\lesssim (\log\langle T\rangle)^2 VIII_{k-1} \cdot I_k \cdot VIII_k.
\end{aligned}$$

For the seventh term in (2.9), we apply (3.1) (and (3.3)), which yields

$$\begin{aligned}
&\int\int_{C_\tau^R} r\Big(|(\partial_t + \partial_r)\partial^{\le 1} u_{k-1}| + \frac{|\partial^{\le 1} u_{k-1}|}{r}\Big)|\partial Z^{\le 60} u_k|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\lesssim \Big(\|\not\partial Z^{\le 5} u_{k-1}\|_{L^2L^2(\tilde C_\tau^R)} + R\|\not\partial\partial Z^{\le 4} u_{k-1}\|_{L^2L^2(\tilde C_\tau^R)} + \|r^{-1} Z^{\le 5} u_{k-1}\|_{L^2L^2(\tilde C_\tau^R)}\Big) \\
&\quad \times \|\langle r\rangle^{-\frac12}\partial Z^{\le 60} u_k\|^2_{L^2L^2(C_\tau^R)}.
\end{aligned}$$

On the C_τ^U regions, in addition to (3.2), we will also apply (3.6) to the term that already contains a good derivative, which gives

$$\begin{aligned}
&\int\int_{C_\tau^U} r\Big(|(\partial_t + \partial_r)\partial^{\le 1} u_{k-1}| + \frac{|\partial^{\le 1} u_{k-1}|}{r}\Big)|\partial Z^{\le 60} u_k|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\lesssim \Big(\frac{1}{U^{\frac12}\tau^{\frac12}}\|Z^{\le 5} u_{k-1}\|_{L^2L^2(\tilde C_\tau^U)} + \frac{1}{U^{\frac12}\tau^{\frac12}}\|(\partial_t + \partial_r)(r Z^{\le 4} u_{k-1})\|_{L^2L^2(\tilde C_\tau^U)} \\
&\quad + \frac{U^{\frac12}}{\tau^{\frac12}}\|\partial(\partial_t + \partial_r)(r Z^{\le 4} u_{k-1})\|_{L^2L^2(\tilde C_\tau^U)}\Big) \\
&\quad \times \|\langle r\rangle^{-\frac12}\partial Z^{\le 60} u_k\|^2_{L^2L^2(C_\tau^U)}.
\end{aligned}$$

Using (1.9), note that

$$\begin{aligned}&\frac{U^{\frac{1}{2}}}{\tau^{\frac{1}{2}}}\|\partial(\partial_t+\partial_r)(rZ^{\leq 4}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\\&\quad\lesssim\frac{U^{\frac{1}{2}}}{\tau^{\frac{1}{2}}}\|\partial Z^{\leq 4}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}+U^{\frac{1}{2}}\tau^{\frac{1}{2}}\|\not\partial\partial Z^{\leq 4}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}.\end{aligned}$$

Thus, upon summing and using (4.12), we obtain

$$\begin{aligned}&\int_0^T\int r\Big(|(\partial_t+\partial_r)\partial^{\leq 1}u_{k-1}|+\frac{|\partial^{\leq 1}u_{k-1}|}{r}\Big)|\partial Z^{\leq 60}u_k|^2\,dx\,dt\\&\lesssim(III_{k-1}+X_{k-1}+IV_{k-1})(\log\langle T\rangle)^{\frac{1}{2}}II_k^2\\&\quad+\Big[VII_{k-1}+(\log\langle T\rangle)\Big(VIII_{k-1}+XI_{k-1}+XII_{k-1}\Big)\Big]\log\langle T\rangle II_k^2.\end{aligned}\tag{4.52}$$

The desired bound (4.16) now follows from (2.9) and the application of (1.5), (4.18), (4.19), (4.49), (4.50), (4.51), (4.52), and (4.6).
[IX$_k$]: As $R<\tau/2$ on each C_τ^R, (3.5) allows us to see that

$$IX_k^2\lesssim\sum_{\tau\leq T}\sum_{R\leq\tau/4}\Big(\frac{1}{R^2}\|Z^{\leq 51}u_k\|^2_{L^2L^2(\tilde{C}_\tau^R)}+\|\not\partial Z^{\leq 50}u_k\|^2_{L^2L^2(\tilde{C}_\tau^R)}\Big)\lesssim III_k^2+IV_k^2.\tag{4.53}$$

And, thus, the appropriate bound is a consequence of (4.16).
[X$_k$]: Using (1.8) and (3.5), when $R<\tau/2$, we see that

$$\begin{aligned}R\|\not\partial\partial Z^{\leq 40}u_k\|_{L^2L^2(\tilde{C}_\tau^R)}&\lesssim\|\partial Z^{\leq 41}u_k\|_{L^2L^2(\tilde{C}_\tau^R)}\\&\quad+R\|(\partial_t+\partial_r)(\partial_t-\partial_r)Z^{\leq 40}u_k\|_{L^2L^2(\tilde{C}_\tau^R)}\\&\lesssim\|\partial Z^{\leq 41}u_k\|_{L^2L^2(\tilde{C}_\tau^R)}+R\|\Box Z^{\leq 40}u_k\|_{L^2L^2(\tilde{C}_\tau^R)}.\end{aligned}\tag{4.54}$$

On $\tilde{C}_\tau^R$, (4.54) is akin to the bounds of [10] and [22]. We note that

$$\begin{aligned}|\Box Z^{\leq 40}u_k|&\lesssim\Big(|\partial Z^{\leq 20}u_{k-1}|+|\partial Z^{\leq 21}u_k|\Big)|Z^{\leq 41}u_{k-1}|\\&\quad+|Z^{\leq 21}u_{k-1}|\Big(|\partial Z^{\leq 40}u_{k-1}|+|\partial Z^{\leq 41}u_k|\Big).\end{aligned}\tag{4.55}$$

Applying (3.1), we obtain

$$\begin{aligned}&R\|Z^{\leq 40}\Box u_k\|_{L^2L^2(\tilde{C}_\tau^R)}\lesssim R^{-1}\|Z^{\leq 41}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\\&\Big(\|\partial Z^{\leq 24}u_{k-1}\|_{L^2L^2(\tilde{\tilde{C}}_\tau^R)}+R\|\not\partial\partial Z^{\leq 23}u_{k-1}\|_{L^2L^2(\tilde{\tilde{C}}_\tau^R)}\Big)\end{aligned}$$

$$+R^{-1}\|Z^{\le 41}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big(\|\partial Z^{\le 25}u_k\|_{L^2L^2(\tilde{\tilde{C}}_\tau^R)}+R\|\not\partial\partial Z^{\le 24}u_k\|_{L^2L^2(\tilde{\tilde{C}}_\tau^R)}\Big)$$
$$+\Big(R^{-1}\|Z^{\le 25}u_{k-1}\|_{L^2L^2(\tilde{\tilde{C}}_\tau^R)}+\|\not\partial Z^{\le 24}u_{k-1}\|_{L^2L^2(\tilde{\tilde{C}}_\tau^R)}\Big)$$
$$\times\Big(\|\partial Z^{\le 40}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}+\|\partial Z^{\le 41}u_k\|_{L^2L^2(\tilde{C}_\tau^R)}\Big).$$

Combining this with (4.54), we see that

$$\begin{aligned}X_k \lesssim\ & IX_k+IV_{k-1}\Big(IX_{k-1}+X_{k-1}+\log\langle T\rangle(XI_{k-1}+XII_{k-1})\Big)\\ &+IV_{k-1}\Big(IX_k+X_k+\log\langle T\rangle(XI_k+XII_k)\Big)\\ &+\Big(IV_{k-1}+III_{k-1}\Big)\Big(IX_{k-1}+IX_k\Big).\end{aligned}\tag{4.56}$$

We note that the occurrences of terms XI and XII are due to the enlargement of $\tilde{C}_\tau^R$ when $R=\tau/4$ and $R=\tau/2$. The tails here can be bounded using the C_τ^U terms when $U=\tau/4$. By (4.16), these terms are also bounded by the right side of (4.3).

[$\mathbf{XI_k}$]: For $U=1$, we have an immediate bound

$$\sum_\tau \frac{1}{\tau(\log\langle\tau\rangle)^2}\|\partial Z^{\le 50}u_k\|^2_{L^2L^2(\tilde{C}_\tau^{U=1})}\lesssim\sum_\tau\frac{1}{(\log\langle\tau\rangle)^2}\|\partial Z^{\le 50}u_k\|^2_{L_t^\infty L_x^2}\le I_k^2.$$

For $U>1$, we can refer to (1.8) and (3.5) to see that

$$\begin{aligned}\frac{U^{\frac12}}{\tau^{\frac12}\log\langle\tau\rangle}\|\partial Z^{\le 50}u_k\|_{L^2L^2(\tilde{C}_\tau^U)}\le\ & \|r^{-\frac12}\langle t-r\rangle^{-\frac12}(\log\langle r\rangle)^{-1}Z^{\le 51}u_k\|_{L^2L^2(\tilde{C}_\tau^U)}\\ &+\|r^{-\frac12}\langle t-r\rangle^{-\frac12}(\partial_t+\partial_r)(rZ^{\le 50}u_k)\|_{L^2L^2(\tilde{C}_\tau^U)}.\end{aligned}$$

And thus,

$$XI_k\lesssim I_k+VIII_k+VII_k.\tag{4.57}$$

Then (4.6) and (4.16) give the boundedness in terms of the right side of (4.3).

[$\mathbf{XII_k}$]: Using (1.8), we first note that

$$\begin{aligned}&\sum_\tau\frac{\tau}{(\log\langle\tau\rangle)^2}\|\not\partial\partial Z^{\le 40}u_k\|^2_{L^2L^2(\tilde{C}_\tau^{U=1})}\\ &\qquad\lesssim\sum_\tau\frac{1}{\tau(\log\langle\tau\rangle)^2}\Big(\|Z^{\le 42}u_k\|^2_{L^2L^2(\tilde{C}_\tau^{U=1})}\\ &\qquad\quad+\|(\partial_t+\partial_r)(rZ^{\le 41}u_k)\|^2_{L^2L^2(\tilde{C}_\tau^{U=1})}\Big)\\ &\qquad\lesssim VIII_k^2+VII_k^2.\end{aligned}$$

And for $1 < U \leq \tau/4$, (1.8) and (3.5) give

$$\begin{aligned} U^{\frac{1}{2}}\tau^{\frac{1}{2}}\|\not\partial\partial Z^{\leq 40}u_k\|_{L^2L^2(\tilde{C}_\tau^U)} &\lesssim \frac{U^{\frac{1}{2}}}{\tau^{\frac{1}{2}}}\|\partial Z^{\leq 41}u_k\|_{L^2L^2(\tilde{C}_\tau^U)} \\ &\quad + U^{\frac{1}{2}}\tau^{\frac{1}{2}}\|(\partial_t+\partial_r)(\partial_t-\partial_r)Z^{\leq 40}u_k\|_{L^2L^2(\tilde{C}_\tau^U)} \\ &\lesssim \frac{U^{\frac{1}{2}}}{\tau^{\frac{1}{2}}}\|\partial Z^{\leq 41}u_k\|_{L^2L^2(\tilde{C}_\tau^U)} \\ &\quad + U^{\frac{1}{2}}\tau^{\frac{1}{2}}\|\Box Z^{\leq 40}u_k\|_{L^2L^2(\tilde{C}_\tau^U)}. \end{aligned}$$

Applying (3.2) to the lower order terms of (4.55) gives

$$\begin{aligned} &U^{\frac{1}{2}}\tau^{\frac{1}{2}}\|\Box Z^{\leq 40}u_k\|_{L^2L^2(\tilde{C}_\tau^U)} \\ &\lesssim \frac{1}{U^{\frac{1}{2}}\tau^{\frac{1}{2}}}\|Z^{\leq 41}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}\Big(\frac{U^{\frac{1}{2}}}{\tau^{\frac{1}{2}}}\|\partial Z^{\leq 24}u_{k-1}\|_{L^2L^2(\tilde{\tilde{C}}_\tau^U)} \\ &\quad + U^{\frac{1}{2}}\tau^{\frac{1}{2}}\|\not\partial\partial Z^{\leq 23}u_{k-1}\|_{L^2L^2(\tilde{\tilde{C}}_\tau^U)}\Big) \\ &\quad + \frac{1}{U^{\frac{1}{2}}\tau^{\frac{1}{2}}}\|Z^{\leq 41}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}\Big(\frac{U^{\frac{1}{2}}}{\tau^{\frac{1}{2}}}\|\partial Z^{\leq 25}u_k\|_{L^2L^2(\tilde{\tilde{C}}_\tau^U)} \\ &\quad + U^{\frac{1}{2}}\tau^{\frac{1}{2}}\|\not\partial\partial Z^{\leq 24}u_k\|_{L^2L^2(\tilde{\tilde{C}}_\tau^U)}\Big) \\ &\quad + \frac{1}{U^{\frac{1}{2}}\tau^{\frac{1}{2}}}\Big(\|Z^{\leq 25}u_{k-1}\|_{L^2L^2(\tilde{\tilde{C}}_\tau^U)} + \|(\partial_t+\partial_r)(rZ^{\leq 24}u_{k-1})\|_{L^2L^2(\tilde{\tilde{C}}_\tau^U)}\Big) \\ &\quad \times \frac{U^{\frac{1}{2}}}{\tau^{\frac{1}{2}}}\Big(\|\partial Z^{\leq 40}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \|\partial Z^{\leq 41}u_k\|_{L^2L^2(\tilde{C}_\tau^U)}\Big). \end{aligned}$$

From this, it follows that

$$\begin{aligned} XII_k \lesssim\ & VIII_k + VII_k + XI_k + VIII_{k-1} \\ & \Big(IX_{k-1} + X_{k-1} + \log\langle T\rangle\Big(XI_{k-1} + XII_{k-1}\Big)\Big) \\ & + VIII_{k-1}\Big(IX_k + X_k + \log\langle T\rangle\Big(XI_k + XII_k\Big)\Big) \\ & + \Big(\log\langle T\rangle VIII_{k-1} + VII_{k-1}\Big)\Big(XI_{k-1} + XI_k\Big). \end{aligned} \tag{4.58}$$

The proof of (4.3) is now complete as we can apply (4.16) and (4.57).

4.2 Convergence

We now conclude the proof by showing that the sequence is Cauchy. This is somewhat simplified by the extremely high regularity that was used in the preceding subsection.

We set

$$A_k = \|\partial Z^{\leq 20}(u_k - u_{k-1})\|_{L^\infty L^2} + \|Z^{\leq 20}(u_k - u_{k-1})\|_{LE^1}, \tag{4.59}$$

and we shall show that for each k,

$$A_k \leq \frac{1}{2} A_{k-1}, \tag{4.60}$$

which suffices to complete the proof.

We use that

$$\begin{aligned}
\Box u_k - \Box u_{k-1} &= a_{JK}^{I,\alpha}(u_{k-1}^J - u_{k-2}^J)\partial_\alpha u_{k-1}^K + a_{JK}^{I,\alpha} u_{k-2}^J \partial_\alpha (u_{k-1}^K - u_{k-2}^K) \\
&+ b_{JK}^{I,\alpha\beta}\partial_\alpha(u_{k-1}^J - u_{k-2}^J)\partial_\beta u_{k-1}^K + b_{JK}^{I,\alpha\beta}\partial_\alpha u_{k-2}^J \partial_\beta (u_{k-1}^K - u_{k-2}^K) \\
&+ A_{JK}^{I,\alpha\beta} u_{k-1}^K \partial_\alpha\partial_\beta(u_k^J - u_{k-1}^J) + A_{JK}^{I,\alpha\beta}(u_{k-1}^K - u_{k-2}^K)\partial_\alpha\partial_\beta u_{k-1}^J \\
&+ B_{JK}^{I,\alpha\beta\gamma}\partial_\gamma u_{k-1}^K \partial_\alpha\partial_\beta(u_k^J - u_{k-1}^J) + B_{JK}^{I,\alpha\beta\gamma}\partial_\gamma(u_{k-1}^K - u_{k-2}^K)\partial_\alpha\partial_\beta u_{k-1}^J.
\end{aligned} \tag{4.61}$$

With h as in (4.5), we apply (2.2). Arguing as in (4.9), though we need not take as much care to distinguish the lower order terms from the higher order terms, this gives that

$$\begin{aligned}
A_k^2 \lesssim & \int_0^T \int |\partial^{\leq 1} Z^{\leq 20}(u_{k-1} - u_{k-2})||\partial Z^{\leq 21} u_{k-1}| \\
& \left(|\partial Z^{\leq 20}(u_k - u_{k-1})| + \frac{|Z^{\leq 20}(u_k - u_{k-1})|}{r}\right) \mathrm{d}x\,\mathrm{d}t \\
& + \int_0^T \int |\partial^{\leq 1} Z^{\leq 20} u_{k-2}||\partial Z^{\leq 20}(u_{k-1} - u_{k-2})| \\
& \left(|\partial Z^{\leq 20}(u_k - u_{k-1})| + \frac{|Z^{\leq 20}(u_k - u_{k-1})|}{r}\right) \mathrm{d}x\,\mathrm{d}t \\
& + \int_0^T \int |\partial^{\leq 1} Z^{\leq 20} u_{k-1}||\partial Z^{\leq 20}(u_k - u_{k-1})| \\
& \left(|\partial Z^{\leq 20}(u_k - u_{k-1})| + \frac{|Z^{\leq 20}(u_k - u_{k-1})|}{r}\right) \mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.62}$$

Applying (3.1) and (3.3), we obtain

$$\int\int_{C_\tau^R} |\partial^{\le 1} Z^{\le 20}(u_{k-1}-u_{k-2})||\partial Z^{\le 21}u_{k-1}|$$
$$\Big(|\partial Z^{\le 20}(u_k-u_{k-1})| + \frac{|Z^{\le 20}(u_k-u_{k-1})|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$\lesssim \Big(\|\partial Z^{\le 25}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + R\|\partial\!\!\!/\partial Z^{\le 24}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big)$$
$$\|\langle r\rangle^{-\frac{3}{2}}\partial^{\le 1}Z^{\le 20}(u_{k-1}-u_{k-2})\|_{L^2L^2(C_\tau^R)}$$
$$\times\Big(\|\langle r\rangle^{-\frac{1}{2}}\partial Z^{\le 20}(u_k-u_{k-1})\|_{L^2L^2(C_\tau^R)} + \|\langle r\rangle^{-\frac{1}{2}}r^{-1}Z^{\le 20}(u_k-u_{k-1})\|_{L^2L^2(C_\tau^R)}\Big).$$

And (3.2) gives

$$\int\int_{C_\tau^U} |\partial^{\le 1} Z^{\le 20}(u_{k-1}-u_{k-2})||\partial Z^{\le 21}u_{k-1}|$$
$$\Big(|\partial Z^{\le 20}(u_k-u_{k-1})| + \frac{|Z^{\le 20}(u_k-u_{k-1})|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$\lesssim \frac{U^{\frac{1}{2}}}{\tau^{\frac{1}{2}}}\Big(\|\partial Z^{\le 25}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \tau\|\partial\!\!\!/\partial Z^{\le 24}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)$$
$$\|\langle t-r\rangle^{-1}\langle r\rangle^{-\frac{1}{2}}\partial^{\le 1}Z^{\le 20}(u_{k-1}-u_{k-2})\|_{L^2L^2(C_\tau^U)}$$
$$\times\Big(\|\langle r\rangle^{-\frac{1}{2}}\partial Z^{\le 20}(u_k-u_{k-1})\|_{L^2L^2(C_\tau^U)} + \|\langle r\rangle^{-\frac{1}{2}}r^{-1}Z^{\le 20}(u_k-u_{k-1})\|_{L^2L^2(C_\tau^U)}\Big).$$

We may use (2.11) and (4.12) to see that

$$\|\langle t-r\rangle^{-1}\langle r\rangle^{-\frac{1}{2}}\partial^{\le 1}Z^{\le 20}(u_{k-1}-u_{k-2})\|_{L^2L^2}$$
$$\lesssim (\log\langle T\rangle)^{\frac{1}{2}}\|Z^{\le 20}(u_{k-1}-u_{k-2})\|_{LE^1}.$$

Thus, upon summing over $R\le\tau/2$, $U\le\tau/4$ and $\tau\le T$, we see that

$$\int_0^T\int |\partial^{\le 1} Z^{\le 20}(u_{k-1}-u_{k-2})||\partial Z^{\le 21}u_{k-1}|$$
$$\Big(|\partial Z^{\le 20}(u_k-u_{k-1})| + \frac{|Z^{\le 20}(u_k-u_{k-1})|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$\lesssim (\log\langle T\rangle)^2 M_{k-1}A_{k-1}A_k. \tag{4.63}$$

Applying (3.1) (and (3.3)) and (3.2), respectively, gives

$$\int\int_{C_\tau^R} |\partial^{\le 1} Z^{\le 20}u_{k-2}||\partial Z^{\le 20}(u_{k-1}-u_{k-2})|$$
$$\Big(|\partial Z^{\le 20}(u_k-u_{k-1})| + \frac{|Z^{\le 20}(u_k-u_{k-1})|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$\lesssim \Big(\|r^{-1}Z^{\le 25}u_{k-2}\|_{L^2L^2(\tilde{C}_\tau^R)} + \|\partial\!\!\!/ Z^{\le 24}u_{k-2}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big)$$

$$\|\langle r\rangle^{-\frac{1}{2}}\partial Z^{\leq 20}(u_{k-1}-u_{k-2})\|_{L^2L^2(C_\tau^R)}$$
$$\times\Big(\|\langle r\rangle^{-\frac{1}{2}}\partial Z^{\leq 20}(u_k-u_{k-1})\|_{L^2L^2(C_\tau^R)}+\|\langle r\rangle^{-\frac{1}{2}}r^{-1}Z^{\leq 20}(u_k-u_{k-1})\|_{L^2L^2(C_\tau^R)}\Big)$$

and

$$\begin{aligned}
&\int\int_{C_\tau^U}|\partial^{\leq 1}Z^{\leq 20}u_{k-2}||\partial Z^{\leq 20}(u_{k-1}-u_{k-2})|\\
&\quad\Big(|\partial Z^{\leq 20}(u_k-u_{k-1})|+\frac{|Z^{\leq 20}(u_k-u_{k-1})|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim\frac{1}{U^{\frac{1}{2}}\tau^{\frac{1}{2}}}\Big(\|Z^{\leq 25}u_{k-2}\|_{L^2L^2(\tilde{C}_\tau^U)}+\|(\partial_t+\partial_r)(rZ^{\leq 24}u_{k-2})\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)\\
&\quad\|\langle r\rangle^{-\frac{1}{2}}\partial Z^{\leq 20}(u_{k-1}-u_{k-2})\|_{L^2L^2(C_\tau^U)}\\
&\quad\times\Big(\|\langle r\rangle^{-\frac{1}{2}}\partial Z^{\leq 20}(u_k-u_{k-1})\|_{L^2L^2(C_\tau^U)}+\|\langle r\rangle^{-\frac{1}{2}}r^{-1}Z^{\leq 20}(u_k-u_{k-1})\|_{L^2L^2(C_\tau^U)}\Big),
\end{aligned}$$

which, using (4.12), imply

$$\begin{aligned}
&\int_0^T\int|\partial^{\leq 1}Z^{\leq 20}u_{k-2}||\partial Z^{\leq 20}(u_{k-1}-u_{k-2})|\\
&\quad\Big(|\partial Z^{\leq 20}(u_k-u_{k-1})|+\frac{|Z^{\leq 20}(u_k-u_{k-1})|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim(\log\langle T\rangle)^2M_{k-2}A_{k-1}A_k.
\end{aligned}\tag{4.64}$$

The third term in (4.62) is very much of the same from as the preceding term, and the exact same arguments yield

$$\begin{aligned}
&\int_0^T\int|\partial^{\leq 1}Z^{\leq 20}u_{k-1}||\partial Z^{\leq 20}(u_k-u_{k-1})|\\
&\quad\Big(|\partial Z^{\leq 20}(u_k-u_{k-1})|+\frac{|Z^{\leq 20}(u_k-u_{k-1})|}{r}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\lesssim(\log\langle T\rangle)^2M_{k-1}A_k^2.
\end{aligned}\tag{4.65}$$

It now follows from (4.62), (4.63), (4.64), and (4.65) that

$$A_k^2\lesssim(\log\langle T\rangle)^2(M_{k-1}+M_{k-2})A_{k-1}A_k+(\log\langle T\rangle)^2M_{k-1}A_k^2.$$

Using (4.3) and (1.6), provided $c\ll 1$, we may bootstrap and obtain

$$A_k^2\lesssim c^4\varepsilon^{\frac{2}{3}}A_{k-1}^2.$$

Thus, for ε sufficiently small, we recover (4.60), which implies that the sequence is Cauchy and thus convergent. This completes the proof.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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