AN r^p-WEIGHTED LOCAL ENERGY APPROACH TO GLOBAL EXISTENCE FOR NULL FORM SEMILINEAR WAVE EQUATIONS

MICHAEL FACCI, ALEX MCENTARFFER, AND JASON METCALFE

Abstract. In this article, we revisit the proof of small data global existence for semilinear wave equations that satisfy a null condition. This new approach relies on a weighted local energy estimate that is akin to those of Dafermos and Rodnianski. Using weighted Sobolev estimates to obtain spatial decay and arguing in the spirit of the work of Keel, Smith, and Sogge, we are able to obtain global existence while only relying on translational and (spatial) rotational symmetries.

1. Introduction

We shall examine systems of semilinear wave equations in $(1+3)$-dimensions of the form

$$\begin{cases}\Box u^I := (\partial_t^2-\Delta)u^I = Q^I(\partial u), \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3, \quad I=1,2,\dots,M,\\ u^I(0,\,\cdot\,)=f^I, \quad \partial_t u^I(0,\,\cdot\,)=g^I.\end{cases} \tag{1.1}$$

Here $\partial u = (\partial_t u, \nabla u)$ is the space-time gradient, and each component Q^I is a smooth function that vanishes to second order at the origin. As we shall only consider small data, the long-time behavior is dictated by the lowest order terms, and as such, we will truncate Q to the quadratic level.

As the linear wave equation decays like $t^{-(n-1)/2}$ in n-spatial dimensions and as this factor is integrable at infinity when $n \geq 4$, it has long been known that global existence of solutions to (1.1) for sufficiently small initial data is guaranteed in these dimensions. When $n = 3$, however, a logarithmic blow up is instead encountered, and only almost global existence, which states that the lifespan of the solution grows exponentially as the size of the initial data shrinks, is available generically. See, e.g., [20].

When the nonlinearity is assumed to satisfy a null condition, it was discovered in [2] and [7] that sufficiently small initial data always produce global solutions in three dimensions. In the current setting, assuming that our quadratic nonlinearity is of the form

$$Q^I(\partial u) = A^{\alpha\beta,I}_{JK}\partial_\alpha u^J \partial_\beta u^K,$$

the null condition requires that

$$A^{\alpha\beta,I}_{JK}\xi_\alpha\xi_\beta = 0, \quad \text{when } \xi_0^2-\xi_1^2-\xi_2^2-\xi_3^2=0. \tag{1.2}$$

Here we are using the summation convention with α,β running from 0 to 3 and the common conventions that $\partial_0 u = \partial_t u$, $\partial_j u = \partial_{x_j} u$. We are also allowing repeated capital indices to sum from 1 to M.

A common approach for establishing such long-time existence results relies on the method of invariant vector fields and the Klainerman-Sobolev inequality [8]. Due to the unbounded normal component on the boundary, the Lorentz boosts $x_k\partial_t + t\partial_k$ are inappropriate when studying such nonlinear equations, say, exterior to a compact obstacle with Dirichlet boundary conditions. In response, [5] developed a method of establishing long-time existence for three dimensional semilinear wave equations that only relies upon the generators of translations and spatial rotations:

$$\Omega_{ij} = x_i\partial_j - x_j\partial_i, \quad Z = (\partial_1,\partial_2,\partial_3,\Omega_{23},\Omega_{13},\Omega_{12}).$$

Here the authors depended on the integrated local energy estimate, which will be introduced in Section 2, and a weighted Sobolev estimate [7] that provided decay in $|x|$ rather than t but only requires the vector

2020 *Mathematics Subject Classification.* 35L71, 35L05.

Key words and phrases. Semilinear wave equations, null condition, global existence, local energy estimate.

The third author gratefully acknowledges the support of a Simons Foundation Collaboration Grant (711724) and National Science Foundation grants DMS-2054910 and DMS-2135998.

fields Z. This method was adapted to the quasilinear setting in [13] by exploring local energy estimates for perturbations of the d'Alembertian. The desire for a method that did not necessitate the use of the Lorentz boosts was also motivated by wanting to understand multiple speed systems of wave equations and the equations of elasticity. See, e.g., [9], [18].

In the current article we shall explore small data global existence for null form wave equations. Many approaches exist for establishing such global existence. See, e.g., [7], [2], [19], [14], [4], [10]. Unlike many of the preceding results, our method shall only rely on the time-independent vector fields Z.

The key to our argument is to replace the use of the local energy estimate with a variant, specifically a type of r^p-weighted local energy estimate of [3]. See [17] for some generalizations of this method. This estimate has been applied in a number of nonlinear settings such as [12], [22, 23], [6] . Typically it is used to derive decay in t. Such decay is then used to control the integral within the energy inequality and thus provides long-time existence. We believe our approach to be more straightforward, though those preceding results were all in much more complicated settings.

The r^p-weighted local energy estimate only controls the "good" derivatives $\not\partial = (\partial_t + \partial_r, \not\nabla)$ where $\not\nabla = \nabla - \frac{x}{r}\partial_r$ are the angular derivatives. These are the directions that are tangent to the light cone and for which better decay is known. The r^p-weighted estimate is particularly well-suited to null form wave equations as the algebraic cancelation condition (1.2) precisely guarantees that in each quadratic term of $Q(\partial u)$ one of the two factors is a good derivative.

Our main result is:

Theorem 1.1. *Suppose that $f, g \in (C^\infty(\mathbb{R}^3))^M$. And let $0 < p < 1$. Then for any $\varepsilon > 0$ sufficiently small, if*

$$\|(1+r)^{p/2} Z^{\le 10} f\|_{L^2(\mathbb{R}^3)} + \|(1+r)^{p/2} Z^{\le 9} g\|_{L^2(\mathbb{R}^3)} \le \varepsilon, \tag{1.3}$$

then (1.1) *with nonlinearity satisfying* (1.2) *has a unique global solution $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$.*

Here, and throughout, we shall use the abbreviation $Z^{\le N} u = \sum_{|\alpha| \le N} Z^\alpha u$.

In this short article, to keep the exposition as accessible as possible, we have only focused on semilinear equations on Minkowski space. We expect that the argument can readily be extended to, e.g., quasilinear equations and equations on exterior domains, and these topics will be explored subsequently.

Our proof of Theorem 1.1 most resembles [10]. There an alternate local energy estimate that relies upon $t - r$ weights, which is from [11] and [1], was used. In order to achieve the decay in $t - r$, the authors called upon decay estimates of [9], but these in turn required the use of the time-dependent vector fields. The current argument is much more directly reminiscent of [5].

2. Integrated local energy estimates

The integrated local energy estimate first appeared in [16]. Through subsequent refinements, on $\mathbb{R}_+ \times \mathbb{R}^n$, $n \ge 3$, we know that

$$\begin{aligned} \|\partial u\|^2_{L^\infty_t L^2_x} + \sup_{R \ge 1} R^{-1} \|\partial u\|^2_{L^2_t L^2_x(\mathbb{R}_+ \times \{\langle x\rangle \approx R\})} + \sup_{R\ge 1} R^{-3} \|u\|^2_{L^2_t L^2_x(\mathbb{R}_+ \times \{\langle x\rangle \approx R\})} \\ \lesssim \|\partial u(0,\cdot)\|^2_{L^2} + \int_0^\infty \int |\Box u| \Big(|\partial u| + \langle x\rangle^{-1}|u|\Big)\, dx\, dt. \end{aligned} \tag{2.1}$$

The most robust proof of this estimate pairs the equation $\Box u$ with a multiplier of the form $C\partial_t u + \frac{r}{r+R}\partial_r u + \frac{n-1}{2}\frac{1}{r+R} u$ and follows from integration by parts. See, e.g., [21] and [13]. Related estimates are known to hold for stationary, non-trapping perturbations and for sufficiently small non-stationary perturbations. See [15] for a more complete history and the most general results in the non-trapping setting.

Our first task will be to prove the following r^p-weighted estimate, which first appeared in [3].

Proposition 2.1. *Suppose $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$ and that for every T there is R so that $u(t,x) = 0$ for $t \in [0,T]$ and $|x| > R$. Then, for $0 < p < 1$,*

$$\|r^{\frac{p-1}{2}} \not\partial u\|^2_{L^2_t L^2_x} + \|r^{\frac{p-3}{2}} u\|^2_{L^2_t L^2_x} + \sup_t \tilde{E}[u](t) \lesssim \tilde{E}[u](0) + \|r^{\frac{p+1}{2}} \Box u\|^2_{L^2_t L^2_x} \tag{2.2}$$

where

$$\tilde{E}[u](t) = \frac{1}{2}\int r^{p-2}|\not\partial(ru(t,x))|^2\,dx + \frac{p}{2}\int r^{p-2}u^2(t,x)\,dx.$$

The local energy estimate (2.1) has an ℓ^∞-summation over the annuli, which we may take to be dyadic, in the left side. In [5], the difference between this and having ℓ^2-summability accounts for a logarithm, which in turn corresponds to the exponential within the notion of almost global existence. While restricted only to the good directions, the above estimate has the desired ℓ^2-summability, and as such, it will yield global existence so long as the equation permits its application on each term, which the null condition exactly provides.

Proof. For any $0 \le p \le 2$, we first consider

$$\int_0^T \int \Box u \cdot r^p\Big(\partial_t u + \partial_r u + \frac{1}{r}u\Big)\,dx\,dt = \int_0^T \int\int r^p\Big(\partial_t^2 - \partial_r^2 - \not\nabla\cdot\not\nabla\Big)(ru)\Big(\partial_t + \partial_r\Big)(ru)\,d\sigma\,dr\,dt.$$

Using integration by parts and the fact that $[\not\nabla, \partial_r] = \frac{1}{r}\not\nabla$, the right side is

$$= \frac{1}{2}\int_0^T \int\int r^p\Big(\partial_t - \partial_r\Big)\Big[\Big(\partial_t + \partial_r\Big)(ru)\Big]^2\,d\sigma\,dr\,dt + \frac{1}{2}\int_0^T\int\int r^p\Big(\partial_t + \partial_r\Big)|\not\nabla(ru)|^2\,d\sigma\,dr\,dt$$
$$+ \int_0^T \int\int r^{p-1}|\not\nabla(ru)|^2\,d\sigma\,dr\,dt.$$

Further integrating by parts yields

$$\int_0^T \int \Box u \cdot r^p\Big(\partial_t u + \partial_r u + \frac{1}{r}u\Big)\,dx\,dt = \frac{1}{2}\int\int r^p\Big\{\Big[\Big(\partial_t + \partial_r\Big)(ru)\Big]^2 + |\not\nabla(ru)|^2\Big\}\,d\sigma\,dr\Big|_{t=0}^T$$
$$+ \frac{p}{2}\int_0^T\int\int r^{p-1}\Big[\Big(\partial_t + \partial_r\Big)(ru)\Big]^2\,d\sigma\,dr\,dt + \Big(1 - \frac{p}{2}\Big)\int_0^T\int\int r^{p-1}|\not\nabla(ru)|^2\,d\sigma\,dr\,dt. \tag{2.3}$$

For simplicity, we now restrict to $0 \le p < 1$. We then observe that

$$\frac{p}{2}\int_0^T\int\int r^{p-1}\Big[\Big(\partial_t + \partial_r\Big)(ru)\Big]^2\,d\sigma\,dr\,dt = \frac{p}{2}\int_0^T\int r^{p-1}\Big(\partial_t u + \partial_r u\Big)^2 r^2\,d\sigma\,dr\,dt$$
$$+ \frac{p}{2}\int_0^T\int\int r^p\Big(\partial_t + \partial_r\Big)u^2\,d\sigma\,dr\,dt + \frac{p}{2}\int_0^T\int r^{p-1}u^2\,d\sigma\,dr\,dt,$$

which upon a last integration by parts and reverting back to rectangular coordinates gives

$$\frac{p}{2}\int r^{p-2}u^2\,dx\Big|_{t=0}^T + \frac{p}{2}\int_0^T\int r^{p-1}\Big(\partial_t u + \partial_r u\Big)^2\,dx\,dt + \frac{p(1-p)}{2}\int_0^T\int r^{p-3}u^2\,dx\,dt.$$

Making this replacement in (2.3) and applying the Schwarz inequality gives

$$\frac{p}{2}\|r^{\frac{p-1}{2}}(\partial_t + \partial_r)u\|^2_{L^2_tL^2_x} + \frac{2-p}{2}\|r^{\frac{p-1}{2}}\not\nabla u\|^2_{L^2_tL^2_x} + \frac{p(1-p)}{2}\|r^{\frac{p-3}{2}}u\|^2_{L^2_tL^2_x} + \tilde{E}[u](T)$$
$$\le \tilde{E}[u](0) + \|r^{\frac{p+1}{2}}\Box u\|_{L^2_tL^2_x}\Big(\|r^{\frac{p-1}{2}}(\partial_t + \partial_r)u\|_{L^2_tL^2_x} + \|r^{\frac{p-3}{2}}u\|_{L^2_tL^2_x}\Big).$$

Bootstrapping the last factor of the last term completes the proof. We moreover note that the implicit constant is independent of T, and thus we may take the supremum over all T to obtain (2.2). $\square$

In the sequel, we shall require a version of (2.2) that permits the application of the invariant vector fields, which is presented in the next proposition.

Proposition 2.2. *Let $0 < p < 1$ and fix any $N \in \mathbb{N}$. Suppose $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$ and that for every T, there is R so that $u(t,x) = 0$ for $t \in [0,T]$ and $|x| > R$. Then,*

$$\|Z^{\le N}\partial u\|^2_{L^\infty_tL^2_x} + \|(1+r)^{\frac{p-1}{2}}Z^{\le N}\not\partial u\|^2_{L^2_tL^2_x} + \|(1+r)^{\frac{p-3}{2}}Z^{\le N}u\|^2_{L^2_tL^2_x}$$
$$\lesssim \|(1+r)^{\frac{p}{2}}Z^{\le N}\partial u(0,\cdot)\|^2_{L^2} + \|(1+r)^{\frac{p+1}{2}}Z^{\le N}\Box u\|^2_{L^2_tL^2_x}. \tag{2.4}$$

Proof. We first note that

$$\int_0^\infty \int |\Box u|\Big(|\partial u| + \langle x\rangle^{-1}|u|\Big)\,dx\,dt \le \|(1+r)^{\frac{p+1}{2}}\Box u\|_{L^2_tL^2_x}\Big(\|(1+r)^{\frac{-p-1}{2}}\partial u\|_{L^2_tL^2_x} + \|(1+r)^{\frac{-p-3}{2}} u\|_{L^2_tL^2_x}\Big),$$

and that

$$\|(1+r)^{\frac{-p-1}{2}}\partial u\|_{L^2_tL^2_x} + \|(1+r)^{\frac{-p-3}{2}} u\|_{L^2_tL^2_x} \lesssim \sup_{j\ge 0} 2^{-j/2}\|\partial u\|_{L^2_tL^2_x(\mathbb{R}_+\times\{\langle x\rangle\approx 2^j\})} + \sup_{j\ge 0} 2^{-3j/2}\|u\|_{L^2_tL^2_x(\mathbb{R}_+\times\{\langle x\rangle\approx 2^j\})}.$$

Thus, by bootstrapping this factor into the left side of (2.1), we see from (2.1) that

$$\|\partial u\|_{L^\infty_t L^2_x} \lesssim \|\partial u(0,\,\cdot\,)\|_{L^2} + \|(1+r)^{\frac{p+1}{2}}\Box u\|_{L^2_tL^2_x}.$$

Since $[\Box, Z]=0$ and since $[Z,\partial]\in \operatorname{span}(\partial)$, the bound for the first term in (2.4) follows by replacing u by $Z^{\le N}u$.

Since

$$[\partial_i, \partial_t+\partial_r] = \frac{1}{r}\nabla\!\!\!/_i,\quad [\partial_i, \nabla\!\!\!/_j] = \frac{1}{r}\Big(-\delta_{ij} + \frac{x_ix_j}{r^2}\Big)\partial_r - \frac{1}{r}\frac{x_j}{r}\nabla\!\!\!/_i,$$
$$[\Omega_{ij}, \nabla\!\!\!/_k] = \delta_{jk}\nabla\!\!\!/_i - \delta_{ik}\nabla\!\!\!/_j,\quad [\Omega_{ij}, \partial_t+\partial_r]=0$$

and since $|\nabla\!\!\!/ u|\le \frac{1}{r}|\Omega u|$, we have that $|[Z,\partial\!\!\!/]u|\le \frac{1}{r}|Zu|$. Thus the remainder of the proof follows upon replacing u by $Z^{\le N}u$ in (2.2). We may readily replace r by $1+r$ in the $L^2_tL^2_x$ terms since the powers in the left are negative while powers in the right are positive. We also note that, due to a Hardy-type inequality,

$$\tilde{E}[u](t) \lesssim \|(1+r)^{\frac{p}{2}}\partial u(t,\,\cdot\,)\|^2_{L^2}.$$

$\square$

3. Proof of Theorem 1.1

The decay that we require will be obtained from the following weighted Sobolev estimate of [7]. This estimate only provides decay in $|x|$, but simultaneously it does not necessitate the use of any time dependent vector fields.

Lemma 3.1. *For $h\in C^\infty(\mathbb{R}^3)$ and $R\ge 1$,*

$$\|h\|_{L^\infty(\{R/2<\langle x\rangle<R\})} \lesssim R^{-1}\|Z^{\le 2}h\|_{L^2(\{R/4<\langle x\rangle<2R\})}. \tag{3.1}$$

The bound (3.1) follows, after localizing appropriately, from applying Sobolev estimates in the r and ω variables separately and comparing the volume element $dr\,d\sigma(\omega)$ with that of $\mathbb{R}^3$ in spherical coordinates: $r^2\,dr\,d\sigma(\omega)$.

As mentioned earlier, the null condition (1.2) guarantees that at least one of the two factors in each nonlinear term is a "good" derivative. In fact, using a product rule argument, we have

$$|Z^{\le 10}Q(\partial u)| \lesssim |Z^{\le 5}\partial u||Z^{\le 10}\partial\!\!\!/ u| + |Z^{\le 5}\partial\!\!\!/ u||Z^{\le 10}\partial u|. \tag{3.2}$$

This is well known, and we refer the reader to, e.g., [10, Lemma 2.3].

We will use an iteration to solve (1.1). We let $u_{-1}\equiv 0$ and let u_k solve

$$\begin{cases} \Box u_k = Q(\partial u_{k-1}), \\ u_k(0,\,\cdot\,) = f, \quad \partial_t u_k(0,\,\cdot\,) = g. \end{cases}$$

Boundedness: Our first step is to show an appropriate boundedness of this iteration. To this end, we shall set

$$M_k = \|Z^{\le 10}\partial u_k\|_{L^\infty_tL^2_x} + \|(1+r)^{\frac{p-1}{2}}Z^{\le 10}\partial\!\!\!/ u_k\|_{L^2_tL^2_x} + \|(1+r)^{\frac{p-3}{2}}Z^{\le 10}u_k\|_{L^2_tL^2_x}.$$

Due to (2.4) and (1.3), there is a constant C_0 so that

$$M_0\le C_0\varepsilon.$$

We shall argue inductively that for every k

$$M_k\le 2C_0\varepsilon. \tag{3.3}$$

To show (3.3), we use (2.4), which provides the bound

$$M_k \leq C_0\varepsilon + C\|(1+r)^{\frac{p+1}{2}} Z^{\leq 10} Q(\partial u_{k-1})\|_{L^2_t L^2_x}.$$

Applying (3.2) and (3.1) we obtain

$$\begin{aligned}
&\|(1+r)^{\frac{p+1}{2}} Z^{\leq 10} Q(\partial u_{k-1})\|_{L^2_t L^2_x} \\
&\qquad\lesssim \|(1+r)^{\frac{p+1}{2}} |Z^{\leq 5}\partial u_{k-1}||Z^{\leq 10}\bar\partial u_{k-1}|\|_{L^2_t L^2_x} + \|(1+r)^{\frac{p+1}{2}} |Z^{\leq 5}\bar\partial u_{k-1}||Z^{\leq 10}\partial u_{k-1}|\|_{L^2_t L^2_x} \\
&\qquad\lesssim \|Z^{\leq 7}\partial u_{k-1}\|_{L^\infty_t L^2_x} \|(1+r)^{\frac{p-1}{2}} Z^{\leq 10}\bar\partial u_{k-1}\|_{L^2_t L^2_x} \\
&\qquad\qquad + \|(1+r)^{\frac{p-1}{2}} Z^{\leq 7}\bar\partial u_{k-1}\|_{L^2_t L^2_x} \|Z^{\leq 10}\partial u_{k-1}\|_{L^\infty_t L^2_x}.
\end{aligned}$$

Thus, using the inductive hypothesis, it follows that

$$M_k \leq C_0\varepsilon + C(M_{k-1})^2 \leq C_0\varepsilon + C\cdot C_0^2\varepsilon^2.$$

And if $\varepsilon < \frac{1}{C\cdot C_0}$, (3.3) results as desired.

Cauchy: We complete the proof by showing that the sequence is Cauchy in an appropriate norm. By completeness, the sequence must converge and by standard results the limiting function solves (1.1) as desired.

To this end, we set

$$A_k = \|Z^{\leq 10}\partial(u_k - u_{k-1})\|_{L^\infty_t L^2_x} + \|(1+r)^{\frac{p-1}{2}} Z^{\leq 10}\bar\partial(u_k - u_{k-1})\|_{L^2_t L^2_x} + \|(1+r)^{\frac{p-3}{2}} Z^{\leq 10}(u_k - u_{k-1})\|_{L^2_t L^2_x}.$$

We note that

$$Q^I(\partial u_{k-1}) - Q^I(\partial u_{k-2}) = A^{\alpha\beta,I}_{JK}\partial_\alpha(u^J_{k-1} - u^J_{k-2})\partial_\beta u^K_{k-1} + A^{\alpha\beta,I}_{JK}\partial_\alpha u^J_{k-2}\partial_\beta(u^K_{k-1} - u^K_{k-2}).$$

Thus, as in (3.2), we obtain

$$\begin{aligned}
(3.4)\quad &|Z^{\leq 10}Q(\partial u_{k-1}) - Z^{\leq 10}Q(\partial u_{k-2})| \lesssim |Z^{\leq 5}\bar\partial(u_{k-1} - u_{k-2})|\Big(|Z^{\leq 10}\partial u_{k-1}| + |Z^{\leq 10}\partial u_{k-2}|\Big) \\
&+ \Big(|Z^{\leq 5}\partial u_{k-1}| + |Z^{\leq 5}\partial u_{k-2}|\Big)|Z^{\leq 10}\bar\partial(u_{k-1} - u_{k-2})| + |Z^{\leq 5}\partial(u_{k-1} - u_{k-2})|\Big(|Z^{\leq 10}\bar\partial u_{k-1}| + |Z^{\leq 10}\bar\partial u_{k-2}|\Big) \\
&\qquad + \Big(|Z^{\leq 5}\bar\partial u_{k-1}| + |Z^{\leq 5}\bar\partial u_{k-2}|\Big)|Z^{\leq 10}\partial(u_{k-1} - u_{k-2})|.
\end{aligned}$$

As above, we apply (3.1) to the lower order factor in each term to see that

$$\begin{aligned}
&\|(1+r)^{\frac{p+1}{2}}(Z^{\leq 10}Q(\partial u_{k-1}) - Z^{\leq 10}Q(\partial u_{k-2}))\|_{L^2_t L^2_x} \\
&\quad\lesssim \|(1+r)^{\frac{p-1}{2}} Z^{\leq 7}\bar\partial(u_{k-1} - u_{k-2})\|_{L^2_t L^2_x}\Big(\|Z^{\leq 10}\partial u_{k-1}\|_{L^\infty_t L^2_x} + \|Z^{\leq 10}\partial u_{k-2}\|_{L^\infty_t L^2_x}\Big) \\
&\quad + \Big(\|Z^{\leq 7}\partial u_{k-1}\|_{L^\infty_t L^2_x} + \|Z^{\leq 7}\partial u_{k-2}\|_{L^\infty_t L^2_x}\Big)\|(1+r)^{\frac{p-1}{2}} Z^{\leq 10}\bar\partial(u_{k-1} - u_{k-2})\|_{L^2_t L^2_x} \\
&+ \|Z^{\leq 7}\partial(u_{k-1} - u_{k-2})\|_{L^\infty_t L^2_x}\Big(\|(1+r)^{\frac{p-1}{2}} Z^{\leq 10}\bar\partial u_{k-1}\|_{L^2_t L^2_x} + \|(1+r)^{\frac{p-1}{2}} Z^{\leq 10}\bar\partial u_{k-2}\|_{L^2_t L^2_x}\Big) \\
&\quad + \Big(\|(1+r)^{\frac{p-1}{2}} Z^{\leq 7}\bar\partial u_{k-1}\|_{L^2_t L^2_x} + \|(1+r)^{\frac{p-1}{2}} Z^{\leq 7}\bar\partial u_{k-2}\|_{L^2_t L^2_x}\Big)\|Z^{\leq 10}\partial(u_{k-1} - u_{k-2})\|_{L^\infty_t L^2_x}.
\end{aligned}$$

From (2.4) it then follows that

$$A_k \leq C\Big(M_{k-1} + M_{k-2}\Big)A_{k-1} \leq C\cdot C_0\varepsilon A_{k-1}.$$

So long as, say, $\varepsilon < \frac{1}{2C\cdot C_0}$, we obtain

$$A_k \leq \frac{1}{2}A_{k-1}, \qquad \text{for all } k,$$

which implies that the sequence is Cauchy and completes the proof.

References

[1] S. Alinhac. The null condition for quasilinear wave equations in two space dimensions I. *Invent. Math.*, 145(3):597–618, 2001.

[2] Demetrios Christodoulou. Global solutions of nonlinear hyperbolic equations for small initial data. *Comm. Pure Appl. Math.*, 39(2):267–282, 1986.

[3] Mihalis Dafermos and Igor Rodnianski. A new physical-space approach to decay for the wave equation with applications to black hole spacetimes. In *XVIth International Congress on Mathematical Physics*, pages 421–432. World Sci. Publ., Hackensack, NJ, 2010.

[4] Soichiro Katayama and Hideo Kubo. An alternative proof of global existence for nonlinear wave equations in an exterior domain. *J. Math. Soc. Japan*, 60(4):1135–1170, 2008.

[5] Markus Keel, Hart F. Smith, and Christopher D. Sogge. Almost global existence for some semilinear wave equations. *J. Anal. Math.*, 87:265–279, 2002. Dedicated to the memory of Thomas H. Wolff.

[6] Joseph Keir. The weak null condition and global existence using the p-weighted energy method. *arXiv preprint arXiv:1808.09982*, 2018.

[7] S. Klainerman. The null condition and global existence to nonlinear wave equations. In *Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984)*, volume 23 of *Lectures in Appl. Math.*, pages 293–326. Amer. Math. Soc., Providence, RI, 1986.

[8] Sergiu Klainerman. Uniform decay estimates and the Lorentz invariance of the classical wave equation. *Comm. Pure Appl. Math.*, 38(3):321–332, 1985.

[9] Sergiu Klainerman and Thomas C. Sideris. On almost global existence for nonrelativistic wave equations in 3D. *Comm. Pure Appl. Math.*, 49(3):307–321, 1996.

[10] Hans Lindblad, Makoto Nakamura, and Christopher D. Sogge. Remarks on global solutions for nonlinear wave equations under the standard null conditions. *J. Differential Equations*, 254(3):1396–1436, 2013.

[11] Hans Lindblad and Igor Rodnianski. Global existence for the Einstein vacuum equations in wave coordinates. *Comm. Math. Phys.*, 256(1):43–110, 2005.

[12] Jonathan Luk. The null condition and global existence for nonlinear wave equations on slowly rotating Kerr spacetimes. *J. Eur. Math. Soc. (JEMS)*, 15(5):1629–1700, 2013.

[13] Jason Metcalfe and Christopher D. Sogge. Long-time existence of quasilinear wave equations exterior to star-shaped obstacles via energy methods. *SIAM J. Math. Anal.*, 38(1):188–209, 2006.

[14] Jason Metcalfe and Christopher D. Sogge. Global existence of null-form wave equations in exterior domains. *Math. Z.*, 256(3):521–549, 2007.

[15] Jason Metcalfe, Jacob Sterbenz, and Daniel Tataru. Local energy decay for scalar fields on time dependent non-trapping backgrounds. *Amer. J. Math.*, 142(3):821–883, 2020.

[16] Cathleen S. Morawetz. Time decay for the nonlinear Klein-Gordon equations. *Proc. Roy. Soc. Ser. A*, 306:291–296, 1968.

[17] Georgios Moschidis. The r^p-weighted energy method of Dafermos and Rodnianski in general asymptotically flat spacetimes and applications. *Ann. PDE*, 2(1):Art. 6, 194, 2016.

[18] Thomas C. Sideris. Nonresonance and global existence of prestressed nonlinear elastic waves. *Ann. of Math. (2)*, 151(2):849–874, 2000.

[19] Thomas C. Sideris and Shu-Yi Tu. Global existence for systems of nonlinear wave equations in 3D with multiple speeds. *SIAM J. Math. Anal.*, 33(2):477–488, 2001.

[20] Christopher D. Sogge. *Lectures on non-linear wave equations*. International Press, Boston, MA, second edition, 2008.

[21] Jacob Sterbenz. Angular regularity and Strichartz estimates for the wave equation. *Int. Math. Res. Not.*, (4):187–231, 2005. With an appendix by Igor Rodnianski.

[22] Shiwu Yang. Global solutions of nonlinear wave equations with large data. *Selecta Math. (N.S.)*, 21(4):1405–1427, 2015.

[23] Shiwu Yang. Global stability of solutions to nonlinear wave equations. *Selecta Math. (N.S.)*, 21(3):833–881, 2015.

Email address: `mfacci@live.unc.edu`
Email address: `mcentarffer@unc.edu`
Email address: `metcalfe@email.unc.edu`

Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250