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On a system of weakly null semilinear wave equations

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Abstract
We develop a new method for addressing certain weakly null systems of wave equations. This approach does not rely on Lorentz invariance nor on the use of null foliations, both of which restrict applications to, e.g., multiple speed systems. The proof uses a class of space-time Klainerman-Sobolev estimates of the first author, Tataru, and Tohaneanu, which pair nicely with local energy estimates that combine the r^p-weighted method of Dafermos and Rodnianski with the ghost weight method of Alinhac. We further refine the standard local energy estimate with a modification of the $\partial_t - \partial_r$ portion of the multiplier.

Keywords Wave equations · Local energy estimates · Weak null condition · Global existence

1 Introduction

This article represents a proof of concept for a method of addressing certain systems of weakly null wave equations that do not satisfy the classical null condition. This example falls into the class of equations studied in [5]. For simplicity of exposition, we only consider a semilinear system. Unlike [5], the methods here do not use the Lorentz boosts, which is important for similar problems in the setting of multiple speeds, exterior domains, or stationary asymptotically flat background geometry. And when compared to the methods of [10], which apply to a broader class of weakly null equations, we believe that our methods are simpler and, as we do not rely on null foliations, additional applications to multiple speeds systems appear possible. The current work is most akin to that of [6], which is based on the ideas of [13] and proves

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global existence without the use of the Lorentz boosts, but we believe our method to have added flexibility for other applications.

In three spatial dimensions, it is known that solutions to semilinear systems of equations of the form $\Box u = Q(\partial u)$ with nonlinearity that vanishes to second order at the origin can only be guaranteed to exist almost globally, which means that the lifespan grows exponentially as the size of the data shrinks. See, e.g., [7] for the lower bound on the lifespan and [8] and [21] for counterexamples to global existence. Based on the fact that the components of the space-time gradient $\partial u = (\partial_t u, \nabla_x u)$ that are tangent to the light cone decay faster, the null condition was identified in [2] and [11] as a sufficient condition for guaranteeing small data global existence. This condition requires that at least one factor of each nonlinear term (at the quadratic level) to be one of the "good" directions. Einstein's equations, for example, do not satisfy this classical null condition, which led to the introduction of the weak null condition in [14, 15] as a possible sufficient condition for small data global existence. Further evidence supporting this is given in [10].

Here we shall consider a coupled system of equations. One of the equations satisfies the classical null condition, but the other does not. The intuition is that the equation satisfying the null condition has a solution that decays faster, and when that is plugged into the second equation, this additional decay allows for an argument to be closed.

We specifically will consider

$$\begin{cases} \Box u = \partial_t u \partial_t v - \nabla u \cdot \nabla v, \\ \Box v = \partial_t v \partial_t u, \\ (u(0,\ \cdot), \partial_t u(0,\ \cdot)) = (u_{(0)}, u_{(1)}), \\ (v(0,\ \cdot), \partial_t v(0,\ \cdot)) = (v_{(0)}, v_{(1)}). \end{cases} \tag{1.1}$$

For simplicity of exposition, we shall take the initial data to be compactly supported, say within $\{|x| \le 2\}$.

In order to describe the "good" directions, we shall frequently orthogonally decompose the (spatial) gradient into radial and angular portions:

$$\nabla = \frac{x}{r}\partial_r + \not\nabla.$$

The directions that are tangent to the light cone are

$$\not\partial = (\partial_t + \partial_r, \not\nabla).$$

By noting that

$$\partial_t u \partial_t v - \nabla u \cdot \nabla v = (\partial_t + \partial_r)u\partial_t v - \partial_r u(\partial_t + \partial_r)v - \not\nabla u \cdot \not\nabla v,$$

we see that the equation for u satisfies the null condition. The equation for v, however, does not. Nevertheless, we shall prove that solutions to (1.1) with sufficiently small initial data exist globally.

Our main theorem is the following statement of global existence.

Theorem 1.1 *Suppose that* $u_{(j)}, v_{(j)} \in C_c^\infty(\mathbb{R}^3)$. *Then there is a* $N \in \mathbb{N}$ *sufficiently large and* $\varepsilon_0 > 0$ *sufficiently small so that if*

$$\sum_{|\alpha|\le N+1} \|\partial_x^\alpha u_{(0)}\|_{L^2} + \sum_{|\alpha|\le N+1} \|\partial_x^\alpha v_{(0)}\|_{L^2} + \sum_{|\alpha|\le N} \|\partial_x^\alpha u_{(1)}\|_{L^2} + \sum_{|\alpha|\le N} \|\partial_x^\alpha v_{(1)}\|_{L^2} \le \varepsilon \tag{1.2}$$

with $\varepsilon \le \varepsilon_0$, *then (1.1) has a unique global solution* $(u, v) \in (C^\infty([0,\infty) \times \mathbb{R}^3))^2$.

The methods that we employ are partly inspired by [9] where almost global existence was established for equations without a null condition by pairing a local energy estimate with a weighted Sobolev estimate that provides decay in $|x|$ rather the t. The latter does not require the use of any time dependent vector fields, which was instrumental in adapting the method of invariant vector fields to, e.g., exterior domains. The paper [3] developed the r^p-weighted local energy estimate. In this variant of the local energy estimate, the additional decay for the "good" derivatives manifests itself as much improved weights. In [4], an analog of [9] was established using these r^p-weighted estimates in order to show global existence for wave equations with the null condition.

In [17], the r^p-weighted multiplier of [3] was combined with a "ghost weight" as in [1]. The resulting estimate allowed for additional improvements on the weight of $(\partial_t + \partial_r)u$ near the light cone. This was then combined with the space-time Klainerman-Sobolev estimates of [20] in order to establish long-time existence for systems of wave equations where the nonlinearity is allowed to depend on the solution not just its derivative. We rely strongly upon these ideas. A further modification of the $(\partial_t - \partial_r)$ component of the multiplier for typical local energy estimates is introduced here. This modification, in particular, while requiring a faster decaying weight also provides a more rapidly decaying weight on the forcing term.

1.1 Notation

Here we fix some notation that will be used throughout the paper. We let

$$\Omega = x \times \nabla, \quad S = t\partial_t + r\partial_r, \quad Z = (\partial_t, \nabla, \Omega, S)$$

denote the admissible vector fields. We will use the shorthand

$$|Z^{\le N}u| = \sum_{|\mu|\le N} |Z^\mu u|, \quad |\partial^{\le N}u| = \sum_{|\mu|\le N} |\partial^\mu u|.$$

A key property of the vector fields Z is that they all preserve solutions to the homogeneous wave equation since

$$[\Box, \partial] = [\Box, \Omega] = 0, \quad [\Box, S] = 2\Box.$$

It will also be important to notice that

$$[Z,\partial]\in \operatorname{span}(\partial),\quad |[Z,\not\partial]u|\le \frac{|Zu|}{r}+|\not\partial u|. \tag{1.3}$$

In the proof of local energy estimates, we will frequently use that

$$[\nabla,\partial_r]=[\not\nabla,\partial_r]=\frac{1}{r}\not\nabla. \tag{1.4}$$

We, moveover, note that

$$\not\nabla=-\frac{x}{r^2}\times\Omega,\quad |\not\nabla u|\le \frac{1}{r}|Zu|. \tag{1.5}$$

We will often decompose $\mathbb{R}^3$ into (inhomogeneous) dyadic regions. To that end, let

$$A_R=\{R\le\langle x\rangle\le 2R\},\quad \tilde{A}_R=\left\{\frac{7}{8}R\le\langle x\rangle\le\frac{17}{8}R\right\}.$$

Similarly, we set

$$X_U=\left\{(t,x)\in\mathbb{R}_+\times\mathbb{R}^3\,:\,U\le\langle t-r\rangle\le 2U\right\},$$

with $\tilde{X}_U$ denoting a similar enlargement.

We shall use a finer refinement, as in [20], when necessary. Because of our assumption that the initial data are supported in $\{|x|\le 2\}$ and because of the finite speed of propagation, it will suffice to examine $C=\{r\le t+2\}$. We then consider a dyadic strip

$$C_\tau=\left\{(t,x)\in\mathbb{R}_+\times\mathbb{R}^3\,:\,\tau\le t\le 2\tau,\ r\le t+2\right\}.$$

Away from the light cone $t=|x|$, we further decompose into dyadic regions in the r variable:

$$C_\tau^{R=1}=C_\tau\cap\{r\le 2\},\quad C_\tau^R=C_\tau\cap\{R\le r\le 2R\}\text{ when } 1<R\le\tau/4.$$

We additionally set

$$\tilde{C}_\tau^{R=1}=C\cap\left\{\frac{7}{8}\tau\le t\le\frac{17}{8}\tau, r\le\frac{17}{8}\right\},$$
$$\tilde{C}_\tau^{R}=C\cap\left\{\frac{7}{8}\tau\le t\le\frac{17}{8}\tau,\ \frac{7}{8}R\le r\le\frac{17}{8}R\right\}\text{ when } 1<R\le\tau/4$$

to denote slight enlargements, which will accommodate the tails of the cutoff functions that are used to localize in the sequel. The key property is that

$$\langle r\rangle \approx R, \quad t-r\approx \tau \quad \text{on } C_\tau^R, \tilde{C}_\tau^R \quad \text{with } \tau \geq 4 \text{ and } 1 \leq R \leq \tau/4.$$

In the vicinity of the light cone, we instead dyadically decompose in $t-|x|$. To this end, let

$$C_\tau^{U=1} = C_\tau \cap \{|t-r| \leq 2\}, \quad C_\tau^U = C_\tau \cap \{U \leq t-r \leq 2U\} \text{ when } 1 < U \leq \tau/4.$$

As above, we denote a slight enlargement on both scales by

$$\tilde{C}_\tau^{U=1} = C \cap \left\{\frac{7}{8}\tau \leq t \leq \frac{17}{8}\tau, |t-r| \leq \frac{17}{8}\right\},$$

and

$$\tilde{C}_\tau^{U} = C \cap \left\{\frac{7}{8}\tau \leq t \leq \frac{17}{8}\tau, \frac{7}{8}U \leq t-r \leq \frac{17}{8}U\right\} \text{ when } 1 < U \leq \tau/4.$$

These choices give

$$r \approx \tau, \quad \langle t-r\rangle \approx U \quad \text{on } C_\tau^U, \tilde{C}_\tau^U \quad \text{with } \tau \geq 4 \text{ and } 1 \leq U \leq \tau/4.$$

With these notations in place, we have

$$C_\tau = \left(\bigcup_{1\leq R\leq \tau/4} C_\tau^R\right) \cup \left(\bigcup_{1\leq U\leq \tau/4} C_\tau^U\right) \cup C_\tau^{\tau/2}$$

where

$$C_\tau^{\tau/2} = C_\tau \cap \{t-r \geq \tau/2\} \cap \{r \geq \tau/2\}.$$

On $C_\tau^{\tau/2}$, we have $r\approx\tau$ and $t-r\approx\tau$. We may regard this region as either a C_τ^R or a C_τ^U region (Fig. 1).

On occasion, we shall use $\tilde{\tilde{C}}_\tau^R, \tilde{\tilde{C}}_\tau^U$ to denote an enlargement of $\tilde{C}_\tau^R, \tilde{C}_\tau^U$ respectively. In the sequel, it will be understood that τ, R, U always run over dyadic values.

In order to localize to such regions, we fix the following notation for cutoff functions. Let χ be a smooth, nonnegative function so that $\chi(z)\equiv 1$ for $z\geq 1$ and $\chi(z)\equiv 0$ for $z\leq 7/8$. We also set

$$\beta(z) = \chi(z) - \chi\left(z-\frac{9}{8}\right)$$

so that $\beta(z)\equiv 1$ for $1\leq z\leq 2$ and $\beta(z)\equiv 0$ when $z\notin[7/8, 17/8]$.

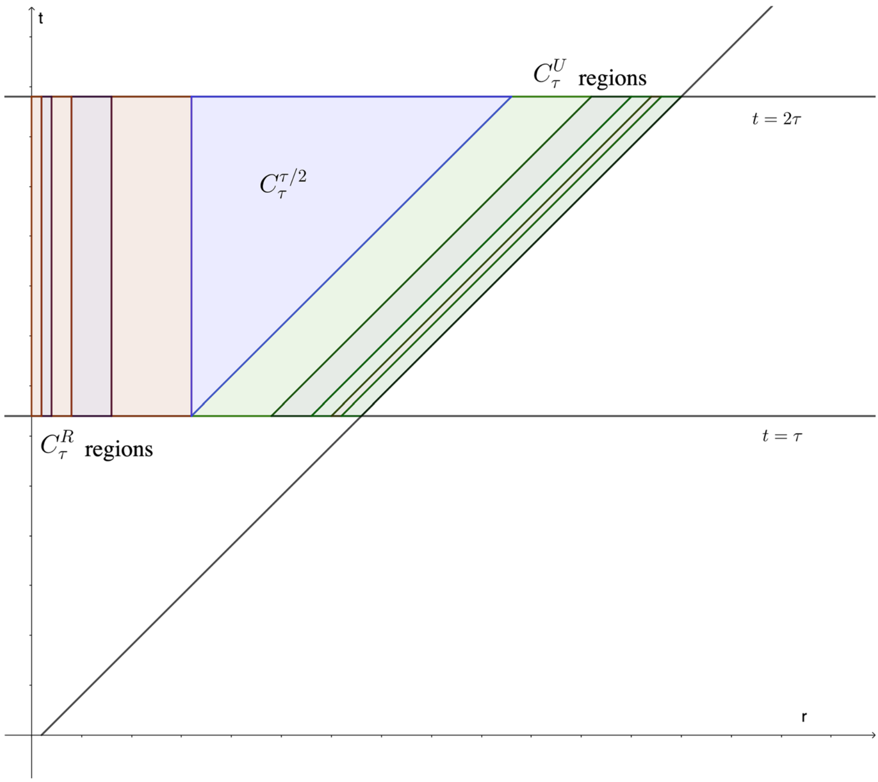

Fig. 1 The decomposition of C_τ into C_τ^R and C_τ^U regions

We will frequently use the mixed norm notation

$$\|u\|_{L^pL^q}^p = \int_0^\infty \|u(t,\,\cdot)\|_{L^q(\mathbb{R}^3)}^p\,dt,$$

with the obvious alteration when $p = \infty$. Unless specified, the domain of all mixed norms of this type is $\mathbb{R}_+ \times \mathbb{R}^3$. We also fix the following local energy norms, which will be discussed more in the next section:

$$\|u\|_{LE} = \sup_{R\geq 1} R^{-\frac{1}{2}}\|u\|_{L^2L^2(\mathbb{R}_+\times A_R)}, \quad \|u\|_{LE^1} = \|(\partial u, u/r)\|_{LE}.$$

2 Local energy estimates

The integrated local energy estimate

$$\|u\|_{LE^1}^2 + \|\partial u\|_{L^\infty L^2}^2 \lesssim \|\partial u(0,\,\cdot)\|_{L^2}^2 + \int_0^\infty \int |\Box u| \left(|\partial u| + \frac{|u|}{r}\right) dx\,dt \quad (2.1)$$

is frequently proved by pairing the equation $\Box u$ with a multiplier of the form $\partial_t + f(r)\partial_r + \frac{f(r)}{r}u$, integrating over a space-time slab, and integrating by parts. The function $f(r)$ needs to be C^2, bounded, non-negative, increasing, and satisfy $-\Delta(f(r)/r) \geq 0$, which the function $f(r) = r/(r+R)$ appropriately satisfies. See [22], [18, 19]. We may rewrite this multiplier as

$$\partial_t + \frac{r}{r+R}\partial_r + \frac{1}{r+R} = \frac{R/2}{r+R}\left(\partial_t - \partial_r - \frac{1}{r}\right) + \frac{r+(R/2)}{r+R}\left(\partial_t + \partial_r + \frac{1}{r}\right), \tag{2.2}$$

which has the property that the coefficient of $\partial_t - \partial_r - \frac{1}{r}$ is nonnegative and decreasing in r, while the coefficient of $\partial_t + \partial_r + \frac{1}{r}$ is nonnegative and increasing. While there are other requirements, this is the key observation to allow for generalizations of the multiplier. In particular, we shall later consider

$$(1+r)^{-\delta}\left(\partial_t - \partial_r - \frac{1}{r}\right) + (1+r)^p e^{-\sigma_U(t-r)}\left(\partial_t + \partial_r + \frac{1}{r}\right).$$

Here $\sigma_U(z) = z/(U+|z|)$, $\delta > 0$, and $0 < p < 2$.

Multipliers of the form $r^p\left(\partial_t + \partial_r + \frac{1}{r}\right)$ appeared previously in [3] and $e^{-\sigma(t-r)}\partial_t$ in [1]. The combination of the two as reflected above is originally from [17]. The change in multiplier on $\partial_t - \partial_r - \frac{1}{r}$ provides an additional degree of decay on the forcing term that helps to close the nonlinear arguments in the sequel.

We first record a corollary of (2.1).

Proposition 2.1 *Suppose $u \in C^2(\mathbb{R}_+ \times \mathbb{R}^3)$ and for all $t \in \mathbb{R}_+$, $|\partial^{\leq 1}u(t,x)| \to 0$ as $|x| \to \infty$. Then*

$$\|u\|_{LE^1} + \|\partial u\|_{L^\infty L^2} \lesssim \|\partial u(0,\cdot)\|_{L^2} + \int_0^\infty \|\Box u(t,\cdot)\|_{L^2}\,dt. \tag{2.3}$$

The proposition follows immediately from (2.1) upon applying the Schwarz inequality to see that

$$\int_0^\infty \int |\Box u|\left(|\partial u| + \frac{|u|}{r}\right)dx\,dt \leq \left(\|\partial u\|_{L^\infty L^2} + \|r^{-1}u\|_{L^\infty L^2}\right)\int_0^\infty \|\Box u(t,\cdot)\|_{L^2}\,dt.$$

A Hardy inequality gives

$$\|r^{-1}u\|_{L^2} \lesssim \|\partial_r u\|_{L^2},$$

which permits the first factor above to be bootstrapped.

We will now discuss the mixed r^p-weighted and ghost weighted estimates of [17], where the former is motivated by [3] and the latter by [1]. To begin, we look at a variant of the Hardy inequality that holds in the space-time norms and yields a "good" derivative. This, in essence, previously appeared in [17].

Lemma 2.2 *Fix $0 < p < 2$. Suppose $u \in C^2(\mathbb{R}_+ \times \mathbb{R}^3)$ and for every $t \in \mathbb{R}_+$, $r^{\frac{p}{2}}|u(t,x)| \to 0$ as $|x| \to \infty$. Then,*

$$\|\langle r\rangle^{\frac{p-1}{2}} r^{-1} u\|_{L^2L^2} + \|\langle r\rangle^{\frac{p-1}{2}} r^{-\frac12} u\|_{L^\infty L^2} \lesssim \|\langle r\rangle^{\frac{p-1}{2}} r^{-\frac12} u(0,\,\cdot)\|_{L^2} + \|\langle r\rangle^{\frac{p-1}{2}} r^{-1}(\partial_t + \partial_r)(ru)\|_{L^2L^2}. \tag{2.4}$$

Proof We write

$$\int_0^T \int \frac{(1+r)^{p-1}}{r^2} u^2\, dx\, dt = -\int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^{p-1} \Big[(\partial_t + \partial_r)(r^{-1})\Big] (ru)^2\, dr\, d\omega\, dt.$$

form on $\mathbb{S}^2$. Integration by parts gives that this is

$$= -\int \frac{(1+r)^{p-1}}{r} u^2(T,x)\, dx + \int \frac{(1+r)^{p-1}}{r} u^2(0,x)\, dx + (p-1)\int_0^T \int \frac{(1+r)^{p-2}}{r} u^2\, dx\, dt + 2\int_0^T \int \frac{(1+r)^{p-1}}{r^2} u(\partial_t + \partial_r)(ru)\, dx\, dt.$$

Using that $\frac{1}{1+r} \le \frac{1}{r}$ in the third term and applying the Schwarz inequality to the last term then shows that

$$(1 - |p-1|)\int_0^T \int \frac{(1+r)^{p-1}}{r^2} u^2\, dx\, dt + \int \frac{(1+r)^{p-1}}{r} u^2(T,x)\, dx \le \int \frac{(1+r)^{p-1}}{r} u^2(0,x)\, dx + 2\Big(\int_0^T \int \frac{(1+r)^{p-1}}{r^2} u^2\, dx\, dt\Big)^{\frac12} \Big(\int_0^T \int (1+r)^{p-1} \Big(r^{-1}(\partial_t + \partial_r)(ru)\Big)^2\, dx\, dt\Big)^{\frac12}.$$

Bootstrapping the first factor of the last term and taking a supremum over T then yields (2.4). □

We next record what, in essence, is the main new estimate of [17].

Proposition 2.3 *Fix $0 < p < 2$. If $u \in C^2(\mathbb{R}_+ \times \mathbb{R}^3)$ and $r^{\frac{p+2}{2}}|\partial^{\le 1} u(t,x)| \to 0$ as $|x| \to \infty$, then*

$$\begin{aligned}
&\|\langle r\rangle^p (\partial_t + \partial_r) u\|_{L^\infty L^2} + \|\langle r\rangle^p \nabla\!\!\!/ u\|_{L^\infty L^2} + \|\langle r\rangle^{\frac{p-1}{2}} r^{-\frac12} u\|_{L^\infty L^2} \\
&\quad + \|\langle r\rangle^{\frac{p-1}{2}} (\partial_t + \partial_r) u\|_{L^2L^2} + \|\langle r\rangle^{\frac{p}{2}} r^{-\frac12} \nabla\!\!\!/ u\|_{L^2L^2} + \|\langle r\rangle^{\frac{p-1}{2}} r^{-1} u\|_{L^2L^2} \\
&\quad + \sup_{U\ge 1} U^{-\frac12} \|\langle r\rangle^{\frac{p}{2}} r^{-1} (\partial_t + \partial_r)(ru)\|_{L^2L^2(X_U)} \\
&\lesssim \|\langle r\rangle^{\frac{p-1}{2}} r^{-\frac12} u(0,\,\cdot)\|_{L^2} + \|\langle r\rangle^{\frac{p}{2}} (\partial_t + \partial_r) u(0,\,\cdot)\|_{L^2} + \|\langle r\rangle^{\frac{p}{2}} \nabla\!\!\!/ u(0,\,\cdot)\|_{L^2} \\
&\quad + \Bigg(\sum_\tau \sum_{R \le \tau/4} \|\langle r\rangle^{\frac{p+1}{2}} \Box u\|^2_{L^2L^2(C^R_\tau)}\Bigg)^{\frac12} + \sum_U \Bigg(\sum_{\tau \ge 4U} U \|\langle r\rangle^{\frac{p}{2}} \Box u\|^2_{L^2L^2(C^U_\tau)}\Bigg)^{\frac12}.
\end{aligned} \tag{2.5}$$

Proof Noting that

$$\Box u = r^{-1}\left(\partial_t^2 - \partial_r^2 - \not\nabla \cdot \not\nabla\right)(ru), \quad \left(\partial_t + \partial_r + \frac{1}{r}\right)u = r^{-1}\left(\partial_t + \partial_r\right)(ru), \tag{2.6}$$

we consider

$$\begin{aligned}&\int_0^T \int (1+r)^p e^{-\sigma_U(t-r)} \Box u \left(\partial_t + \partial_r + \frac{1}{r}\right) u \, dx \, dt \\ &= \int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^p e^{-\sigma_U(t-r)} \left(\partial_t^2 - \partial_r^2 - \not\nabla \cdot \not\nabla\right)(ru)\left(\partial_t + \partial_r\right)(ru) \, dr \, d\omega \, dt\end{aligned}$$

for $0 < p < 2$, which, using (1.4), is equivalent to

$$\begin{aligned}&\frac{1}{2}\int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^p e^{-\sigma_U(t-r)} \left(\partial_t - \partial_r\right)\left[\left(\partial_t + \partial_r\right)(ru)\right]^2 \, dr \, d\omega \, dt \\ &+ \frac{1}{2}\int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^p e^{-\sigma_U(t-r)} \left(\partial_t + \partial_r\right)|\not\nabla(ru)|^2 \, dr \, d\omega \, dt \\ &+ \int_0^T \int_{\mathbb{S}^2} \int_0^\infty \frac{(1+r)^p}{r} e^{-\sigma_U(t-r)} |\not\nabla(ru)|^2 \, dr \, d\omega \, dt.\end{aligned}$$

Subsequent integrations by parts then give

$$\begin{aligned}&\int_0^T \int (1+r)^p e^{-\sigma_U(t-r)} \Box u \left(\partial_t + \partial_r + \frac{1}{r}\right) u \, dx \, dt \\ &= \frac{1}{2}\int_{\mathbb{S}^2} \int_0^\infty (1+r)^p e^{-\sigma_U(t-r)} \left\{\left[\left(\partial_t + \partial_r\right)(ru)\right]^2 + |\not\nabla(ru)|^2\right\} dr \, d\omega \Big|_{t=0}^T \\ &+ \frac{1}{2}\int_0^T \int_{\mathbb{S}^2} e^{-\sigma_U(t)} u^2(t,0) \, d\omega \, dt \\ &+ \frac{p}{2}\int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^{p-1} e^{-\sigma_U(t-r)} \left[\left(\partial_t + \partial_r\right)(ru)\right]^2 \, dr \, d\omega \, dt \\ &+ \int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^p \sigma_U'(t-r) e^{-\sigma_U(t-r)} \left[\left(\partial_t + \partial_r\right)(ru)\right]^2 \, dr \, d\omega \, dt \\ &+ \left(1 - \frac{p}{2}\right)\int_0^T \int_{\mathbb{S}^2} \int_0^\infty \frac{(1+r)^p}{r} e^{-\sigma_U(t-r)} |\not\nabla(ru)|^2 \, dr \, d\omega \, dt \\ &+ \frac{p}{2}\int_0^T \int_{\mathbb{S}^2} \int_0^\infty \frac{(1+r)^{p-1}}{r} e^{-\sigma_U(t-r)} |\not\nabla(ru)|^2 \, dr \, d\omega \, dt.\end{aligned} \tag{2.7}$$

Rearranging the terms, noting that

$$\sigma_U'(t-r) \gtrsim \frac{1}{\langle t-r \rangle}, \quad \text{on } X_U,$$

and taking a supremum over T yields

$$\begin{aligned}
&\|\langle r\rangle^{\frac{p}{2}} r^{-1}(\partial_t+\partial_r)(ru)\|^2_{L^\infty L^2} + \|\langle r\rangle^{\frac{p}{2}} \not\nabla u\|^2_{L^\infty L^2} + \|\langle r\rangle^{\frac{p-1}{2}} r^{-1}(\partial_t+\partial_r)(ru)\|^2_{L^2L^2}\\
&\quad+\|\langle r\rangle^{\frac{p}{2}} r^{-\frac12}\not\nabla u\|^2_{L^2L^2} + \sup_U \|\langle r\rangle^{\frac{p}{2}}\langle t-r\rangle^{-\frac12} r^{-1}(\partial_t+\partial_r)(ru)\|^2_{L^2L^2(X_U)}\\
&\quad\lesssim \|\langle r\rangle^{\frac{p}{2}} r^{-1}u(0,\,\cdot)\|^2_{L^2} + \|\langle r\rangle^{\frac{p}{2}} \not\partial u(0,\,\cdot)\|^2_{L^2} + \int_0^\infty\int \langle r\rangle^p |\Box u||r^{-1}(\partial_t+\partial_r)(ru)|\,dx\,dt.
\end{aligned}$$

By the Schwarz inequality, we may bound

$$\begin{aligned}
&\int_0^\infty\int \langle r\rangle^p |\Box u||r^{-1}(\partial_t+\partial_r)(ru)|\,dx\,dt\\
&\quad\lesssim \Big(\sum_\tau \sum_{R\le\tau/4} \|\langle r\rangle^{\frac{p+1}{2}}\Box u\|^2_{L^2L^2(C_\tau^R)}\Big)^{\frac12} \|\langle r\rangle^{\frac{p-1}{2}} r^{-1}(\partial_t+\partial_r)(ru)\|_{L^2L^2}\\
&\qquad+\Big\{\sum_U \Big[\sum_{\tau\ge 4U} U\|\langle r\rangle^{\frac{p}{2}}\Box u\|^2_{L^2L^2(C_\tau^U)}\Big]^{\frac12}\Big\}\Big(\sup_U U^{-\frac12}\|\langle r\rangle^{\frac{p}{2}} r^{-1}(\partial_t+\partial_r)(ru)\|_{L^2L^2(X_U)}\Big)
\end{aligned}\tag{2.8}$$

The second factor of each term may be bootstrapped. Combining what results with (2.4) completes the proof. □

We next combine the previous proposition with a modification of the $(\partial_t-\partial_r)$ portion of the multiplier in (2.2). While the new $\partial_t-\partial_r$ terms are easily controlled using the LE^1 norm, the corresponding forcing term comes with an added factor of decay, $(1+r)^{-\delta}$, when compared to the right side of (2.1).

Theorem 2.4 *Fix $0<p<2$ and $\delta>0$. If $u\in C^2(\mathbb{R}_+\times\mathbb{R}^3)$ and $r^{\frac{p+2}{2}}|\partial^{\le 1}u(t,x)|\to 0$ as $|x|\to\infty$, then*

$$\begin{aligned}
&\|\langle r\rangle^{-\frac{\delta}{2}}(\partial_t-\partial_r)u\|_{L^\infty L^2} + \|\langle r\rangle^{\frac{p}{2}}(\partial_t+\partial_r)u\|_{L^\infty L^2} + \|\langle r\rangle^{\frac{p}{2}}\not\nabla u\|_{L^\infty L^2}\\
&\quad+\|\langle r\rangle^{\frac{p-1}{2}} r^{-\frac12} u\|_{L^\infty L^2} + \|\langle r\rangle^{-\frac{1+\delta}{2}}(\partial_t-\partial_r)u\|_{L^2L^2} + \|\langle r\rangle^{\frac{p-1}{2}}(\partial_t+\partial_r)u\|_{L^2L^2}\\
&\quad+\|\langle r\rangle^{\frac{p-1}{2}}\not\nabla u\|_{L^2L^2} + \|\langle r\rangle^{\frac{p-1}{2}} r^{-1}u\|_{L^2L^2}\\
&\quad+\sup_U U^{-\frac12}\|\langle r\rangle^{\frac{p}{2}} r^{-1}(\partial_t+\partial_r)(ru)\|_{L^2L^2(X_U)} \lesssim \|\langle r\rangle^{-\frac{\delta}{2}}(\partial_t-\partial_r)u(0,\,\cdot)\|_{L^2}\\
&\quad+\|\langle r\rangle^{\frac{p}{2}}(\partial_t+\partial_r)u(0,\,\cdot)\|_{L^2} + \|\langle r\rangle^{\frac{p}{2}}\not\nabla u(0,\,\cdot)\|_{L^2} + \|\langle r\rangle^{\frac{p}{2}} r^{-1}u(0,\,\cdot)\|_{L^2}\\
&\quad+\|\langle r\rangle^{\frac{1-\delta}{2}}\Box u\|_{L^2L^2} + \Big(\sum_\tau\sum_{R\le\tau/4}\|\langle r\rangle^{\frac{1+p}{2}}\Box u\|^2_{L^2L^2(C_\tau^R)}\Big)^{\frac12}\\
&\quad+\sum_U\Big(\sum_{\tau\ge 4U} U\|\langle r\rangle^{\frac{p}{2}}\Box u\|^2_{L^2L^2(C_\tau^U)}\Big)^{\frac12}.
\end{aligned}\tag{2.9}$$

Proof Using (2.6) and the related identity

$$\left(\partial_t - \partial_r - \frac{1}{r}\right) u = r^{-1} \left(\partial_t - \partial_r\right)(ru),$$

we begin by considering

$$\begin{aligned}
&\int_0^T \int (1+r)^{-\delta} \Box u \left(\partial_t - \partial_r - \frac{1}{r}\right) u \, dx \, dt \\
&= \int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^{-\delta} \left(\partial_t^2 - \partial_r^2 - \not\nabla \cdot \not\nabla\right)(ru)\left(\partial_t - \partial_r\right)(ru) \, dr \, d\omega \, dt
\end{aligned}$$

with $\delta > 0$. Integrating by parts and using (1.4), this is

$$\begin{aligned}
&= \frac{1}{2} \int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^{-\delta} \left(\partial_t + \partial_r\right) \left[\left(\partial_t - \partial_r\right)(ru)\right]^2 \, dr \, d\omega \, dt \\
&\quad - \int_0^T \int_{\mathbb{S}^2} \int_0^\infty \frac{(1+r)^{-\delta}}{r} |\not\nabla(ru)|^2 \, dr \, d\omega \, dt \\
&\quad + \frac{1}{2} \int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^{-\delta} \left(\partial_t - \partial_r\right) |\not\nabla(ru)|^2 \, dr \, d\omega \, dt.
\end{aligned}$$

The Fundamental Theorem of Calculus and subsequent integrations by parts give

$$\begin{aligned}
&\int_0^T \int (1+r)^{-\delta} \Box u \left(\partial_t - \partial_r - \frac{1}{r}\right) u \, dx \, dt \\
&= \frac{1}{2} \int_{\mathbb{S}^2} \int_0^\infty (1+r)^{-\delta} \left\{\left[\left(\partial_t - \partial_r\right)(ru)\right]^2 + |\not\nabla(ru)|^2\right\} dr \, d\omega \Big|_{t=0}^T \\
&\quad - \frac{1}{2} \int_0^T \int_{\mathbb{S}^2} u^2(t,0) \, d\omega \, dt + \frac{\delta}{2} \int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^{-1-\delta} \left[\left(\partial_t - \partial_r\right)(ru)\right]^2 \, dr \, d\omega \, dt \\
&\quad - \int_0^T \int_{\mathbb{S}^2} \int_0^\infty (1+r)^{-\delta} \left(r^{-1} + \frac{\delta}{2}(1+r)^{-1}\right) |\not\nabla(ru)|^2 \, dr \, d\omega \, dt.
\end{aligned} \tag{2.10}$$

We now consider a multiplier of the form

$$(1+r)^{-\delta} \left(\partial_t - \partial_r - \frac{1}{r}\right) + C(1+r)^p e^{-\sigma_U(t-r)} \left(\partial_t + \partial_r + \frac{1}{r}\right), \quad C \gg 1,$$

by adding a large multiple of (2.7) to (2.10). Since σ_U is bounded independently of U, for a sufficiently large C,

$$\frac{C}{2} \int_0^T \int_{\mathbb{S}^2} e^{-\sigma_U(t)} u^2(t,0) \, d\omega \, dt - \frac{1}{2} \int_0^T \int_{\mathbb{S}^2} u^2(t,0) \, d\omega \, dt \geq 0,$$

and as such, this $r = 0$ boundary term may be dropped. The nonnegative contribution

$$\frac{1}{2}\int_{\mathbb{S}^2}\int_0^\infty (1+r)^{-\delta}|\nabla\!\!\!/ (ru)(T, r\omega)|^2\,dr\,d\omega$$

may also be omitted, and since $\delta,\ p > 0$, we can simplify by bounding

$$\frac{1}{2}\int_{\mathbb{S}^2}\int_0^\infty (1+r)^{-\delta}|\nabla\!\!\!/ (ru)(0, r\omega)|^2\,dr\,d\omega \le \frac{1}{2}\int_{\mathbb{S}^2}\int_0^\infty (1+r)^{p}|\nabla\!\!\!/ (ru)(0, r\omega)|^2\,dr\,d\omega.$$

What then results from this combination of (2.10) and (2.7) is

$$\begin{aligned}
&\left\|(1+r)^{-\frac{\delta}{2}}r^{-1}\left(\partial_t-\partial_r\right)(ru)\right\|^2_{L^\infty L^2} \\
&\quad+\left\|(1+r)^{\frac{p}{2}}r^{-1}\left(\partial_t+\partial_r\right)(ru)\right\|^2_{L^\infty L^2}+\left\|(1+r)^{\frac{p}{2}}\nabla\!\!\!/ u\right\|^2_{L^\infty L^2} \\
&\quad+\left\|(1+r)^{-\frac{1}{2}-\frac{\delta}{2}}r^{-1}\left(\partial_t-\partial_r\right)(ru)\right\|^2_{L^2L^2}+\left\|(1+r)^{\frac{p-1}{2}}r^{-1}\left(\partial_t+\partial_r\right)(ru)\right\|^2_{L^2L^2} \\
&\quad+\left\|(1+r)^{\frac{p-1}{2}}\nabla\!\!\!/ u\right\|^2_{L^2L^2} \\
&\quad+\sup_U U^{-1}\left\|(1+r)^{\frac{p}{2}}r^{-1}\left(\partial_t+\partial_r\right)(ru)\right\|^2_{L^2L^2(X_U)} \\
&\lesssim\left\|(1+r)^{-\frac{\delta}{2}}r^{-1}\left(\partial_t-\partial_r\right)(ru)(0,\,\cdot)\right\|^2_{L^2}+\left\|(1+r)^{\frac{p}{2}}r^{-1}\left(\partial_t+\partial_r\right)(ru)(0,\,\cdot)\right\|^2_{L^2} \\
&\quad+\left\|(1+r)^{\frac{p}{2}}\nabla\!\!\!/ u(0,\,\cdot)\right\|^2_{L^2} \\
&\quad+\int_0^\infty\int|\Box u|\left\{(1+r)^{-\delta}\left|\left(\partial_t-\partial_r-\frac{1}{r}\right)u\right|+(1+r)^p\left|\left(\partial_t+\partial_r+\frac{1}{r}\right)u\right|\right\}dx\,dt.
\end{aligned}\tag{2.11}$$

We use (2.8) and the fact that the Schwarz inequality allows us to bound

$$\begin{aligned}
&\int_0^\infty\int|\Box u|(1+r)^{-\delta}\left|\left(\partial_t-\partial_r-\frac{1}{r}\right)u\right|\,dx\,dt \\
&\quad\lesssim\|(1+r)^{\frac{1-\delta}{2}}\Box u\|_{L^2L^2}\|(1+r)^{-\frac{1+\delta}{2}}r^{-1}(\partial_t-\partial_r)(ru)\|_{L^2L^2}.
\end{aligned}$$

Bootstrapping then gives

$$\begin{aligned}
&\|\langle r\rangle^{-\frac{\delta}{2}}r^{-1}(\partial_t-\partial_r)(ru)\|_{L^\infty L^2}+\|\langle r\rangle^{\frac{p}{2}}r^{-1}(\partial_t+\partial_r)(ru)\|_{L^\infty L^2}+\|\langle r\rangle^{\frac{p}{2}}\nabla\!\!\!/ u\|_{L^\infty L^2} \\
&\quad+\|\langle r\rangle^{-\frac{1+\delta}{2}}r^{-1}(\partial_t-\partial_r)(ru)\|_{L^2L^2}+\|\langle r\rangle^{\frac{p-1}{2}}r^{-1}(\partial_t+\partial_r)(ru)\|_{L^2L^2} \\
&\quad+\|\langle r\rangle^{\frac{p-1}{2}}\nabla\!\!\!/ u\|_{L^2L^2} \\
&\quad+\sup_U U^{-\frac{1}{2}}\|\langle r\rangle^{\frac{p}{2}}r^{-1}(\partial_t+\partial_r)(ru)\|_{L^2L^2(X_U)}\lesssim\|\langle r\rangle^{-\frac{\delta}{2}}(\partial_t-\partial_r)u(0,\,\cdot)\|_{L^2} \\
&\quad+\|\langle r\rangle^{\frac{p}{2}}(\partial_t+\partial_r)u(0,\,\cdot)\|_{L^2}+\|\langle r\rangle^{\frac{p}{2}}\nabla\!\!\!/ u(0,\,\cdot)\|_{L^2}+\|\langle r\rangle^{\frac{p}{2}}r^{-1}u(0,\,\cdot)\|_{L^2}
\end{aligned}$$

$$+\|\langle r\rangle^{\frac{1-\delta}{2}}\Box u\|_{L^2L^2}+\left(\sum_{\tau}\sum_{R\leq\tau/4}\|\langle r\rangle^{\frac{1+p}{2}}\Box u\|^2_{L^2L^2(C^R_\tau)}\right)^{\frac{1}{2}}$$

$$+\sum_U\left(\sum_{\tau\geq 4U}U\|\langle r\rangle^{\frac{p}{2}}\Box u\|^2_{L^2L^2(C^U_\tau)}\right)^{\frac{1}{2}}.$$

Pairing this with (2.4) then completes the proof. □

3 Sobolev estimates

In this section, we collect our principal decay estimates, which are variants of the Klainerman-Sobolev estimate [12].

On occasion, it will suffice to apply the following standard weighted Sobolev estimate, which is also from [12] and follows by applying Sobolev embeddings in the r and ω variables after localizing.

Lemma 3.1 *For $h \in C^\infty(\mathbb{R}^3)$ and $R > 0$,*

$$\|h\|_{L^\infty(A_R)} \lesssim R^{-1}\|Z^{\leq 2}h\|_{L^2(\tilde{A}_R)}. \tag{3.1}$$

Where a finer analysis is necessary, we shall use the space-time Klainerman-Sobolev estimates of [20, Lemma 3.8]. We record these in the following lemma.

Lemma 3.2 *If $\tau \geq 1$, $1 \leq R \leq \tau/2$, and $1 \leq U \leq \tau/4$, then*

$$\|w\|_{L^\infty L^\infty(C^R_\tau)} \lesssim \frac{1}{\tau^{\frac{1}{2}}R^{\frac{3}{2}}}\|Z^{\leq 2}w\|_{L^2L^2(\tilde{C}^R_\tau)}+\frac{1}{\tau^{\frac{1}{2}}R^{\frac{1}{2}}}\|\partial_r Z^{\leq 2}w\|_{L^2L^2(\tilde{C}^R_\tau)}, \tag{3.2}$$

$$\|w\|_{L^\infty L^\infty(C^U_\tau)} \lesssim \frac{1}{\tau^{\frac{3}{2}}U^{\frac{1}{2}}}\|Z^{\leq 2}w\|_{L^2L^2(\tilde{C}^U_\tau)}+\frac{U^{\frac{1}{2}}}{\tau^{\frac{3}{2}}}\|\partial_r Z^{\leq 2}w\|_{L^2L^2(\tilde{C}^U_\tau)}. \tag{3.3}$$

We shall only tersely describe the proof since this result previously appeared in [20]. If $R = 1$, (3.2) is an immediate consequence of standard Sobolev embeddings. And if $1 < R \lesssim \tau$, then after localizing, we may apply Sobolev embeddings in (s, ω) and the Fundamental Theorem of Calculus in ρ where $t = e^s$ and $r = e^{s+\rho}$. This gives that

$$\beta\left(\frac{e^s}{\tau}\right)\beta\left(\frac{e^{s+\rho}}{R}\right)|w\left(e^s, e^{s+\rho}\omega\right)|$$
$$\lesssim\left(\int\int\int\left|\partial_\rho\left(\partial^{\leq 2}_{s,\omega}\left[\beta\left(\frac{e^s}{\tau}\right)\beta\left(\frac{e^{s+\rho}}{R}\right)w(e^s, e^{s+\rho}\omega)\right]\right)^2\right|d\rho\, ds\, d\omega\right)^{1/2}.$$

Relying on the observations that

$$\partial_s(w(e^s, e^{s+\rho}\omega)) = (Sw)(e^s, e^{s+\rho}\omega), \quad |\partial_\omega(w(e^s, e^{s+\rho}\omega)| \lesssim |(\Omega w)(e^s, e^{s+\rho}\omega)|,$$
$$\partial_\rho(w(e^s, e^{s+\rho}\omega)) = (r\partial_r w)(e^s, e^{s+\rho}\omega),$$

upon changing variables in the integrals, we see that

$$\|w\|_{L^\infty L^\infty(C^R_\tau)} \lesssim \frac{1}{\tau^{\frac{1}{2}} R^{\frac{3}{2}}} \|Z^{\le 2} w\|_{L^2 L^2(\tilde{C}^R_\tau)} + \frac{1}{\tau^{\frac{1}{2}} R} \|Z^{\le 2} w\|^{\frac{1}{2}}_{L^2 L^2(\tilde{C}^R_\tau)} \|\partial_r Z^{\le 2} w\|^{\frac{1}{2}}_{L^2 L^2(\tilde{C}^R_\tau)}. \tag{3.4}$$

The estimate (3.2) is now an immediate consequence. Moreover, if $R = \tau/2$, if we replace w by $\chi\left(\frac{2(t-r)}{\tau}\right) w$ and note that $S\left(\chi\left(\frac{2(t-r)}{\tau}\right)\right) = \mathcal{O}(1)$, the estimate for $C^{\tau/2}_\tau$ also follows.

When $U = 1$, the bound (3.3) follows from (3.4). Otherwise, with $t = e^s$ and $t - r = e^{s+\rho}$, a similar application of Sobolev embeddings yields (3.3).

When the estimates of [20] are applied to ∂w, the decomposition pairs nicely with

$$2Sw = (t+r)(\partial_t + \partial_r)w + (t-r)(\partial_t - \partial_r)w, \tag{3.5}$$

which will allow us to recover $\Box$ to get additional decay out of the second derivative terms. This represents space-time analogs of some estimates of [13]. See, also, [16] where some similar analyses appeared previously.

Corollary 3.3 *For $\tau \ge 1$ and $1 \le R \le \tau/2$, $1 \le U \le \tau/4$, we have*

$$\|\partial w\|_{L^\infty L^\infty(C^R_\tau)} \lesssim \frac{1}{\tau^{\frac{1}{2}} R^{\frac{3}{2}}} \|\partial Z^{\le 3} w\|_{L^2 L^2(\tilde{C}^R_\tau)} + \frac{1}{\tau^{\frac{1}{2}} R^{\frac{1}{2}}} \|\Box Z^{\le 2} w\|_{L^2 L^2(\tilde{C}^R_\tau)}, \tag{3.6}$$

$$\|\partial w\|_{L^\infty L^\infty(C^U_\tau)} \lesssim \frac{1}{U^{\frac{1}{2}} \tau^{\frac{3}{2}}} \|\partial Z^{\le 3} w\|_{L^2 L^2(\tilde{C}^U_\tau)} + \frac{1}{U^{\frac{1}{2}} \tau^{\frac{1}{2}}} \|\Box Z^{\le 2} w\|_{L^2 L^2(\tilde{C}^U_\tau)}. \tag{3.7}$$

Proof We apply (3.2) to see

$$\|\partial w\|_{L^\infty L^\infty(C^R_\tau)} \lesssim \frac{1}{\tau^{\frac{1}{2}} R^{\frac{3}{2}}} \|\partial Z^{\le 2} w\|_{L^2 L^2(\tilde{C}^R_\tau)} + \frac{1}{\tau^{\frac{1}{2}} R^{\frac{1}{2}}} \|\partial_r \partial Z^{\le 2} w\|_{L^2 L^2(\tilde{C}^R_\tau)}. \tag{3.8}$$

We notice that

$$\|\partial_r \nabla\!\!\!\!/\, Z^{\le 2} w\|_{L^2 L^2(\tilde{C}^R_\tau)} \lesssim \frac{1}{R} \|\partial Z^{\le 3} w\|_{L^2 L^2(\tilde{C}^R_\tau)} \tag{3.9}$$

follows from (1.4) and (1.5). Moreover, by applying (3.5) with w replaced by $(\partial_t - \partial_r)Z^{\le 2} w$ and $(\partial_t + \partial_r)Z^{\le 2} w$ respectively, we obtain

$$\|(\partial_t - \partial_r)^2 Z^{\le 2} w\|_{L^2 L^2(\tilde{C}^R_\tau)} \lesssim \frac{1}{R} \|\partial Z^{\le 3} w\|_{L^2 L^2(\tilde{C}^R_\tau)} + \|(\partial_t^2 - \partial_r^2) Z^{\le 2} w\|_{L^2 L^2(\tilde{C}^R_\tau)}, \tag{3.10}$$

and

$$\|(\partial_t+\partial_r)^2 Z^{\le 2} w\|_{L^2L^2(\tilde{C}^R_\tau)} \lesssim \frac{1}{R}\|\partial Z^{\le 3} w\|_{L^2L^2(\tilde{C}^R_\tau)} + \|(\partial_t^2-\partial_r^2) Z^{\le 2} w\|_{L^2L^2(\tilde{C}^R_\tau)}. \tag{3.11}$$

Observing that

$$|\partial_r \partial Z^{\le 2} w| \le |\partial_r \nabla\!\!\!/ Z^{\le 2} w| + |\partial_r(\partial_t-\partial_r)Z^{\le 2}w| + |\partial_r(\partial_t+\partial_r)Z^{\le 2}w|$$

and subsequently writing $\partial_r = \frac{1}{2}[(\partial_t+\partial_r)-(\partial_t-\partial_r)]$ in the last two terms, we see that

$$\begin{aligned}\|\partial_r \partial Z^{\le 2} w\|_{L^2L^2(\tilde{C}^R_\tau)} \lesssim\ & \|\partial_r \nabla\!\!\!/ Z^{\le 2} w\|_{L^2L^2(\tilde{C}^R_\tau)} + \|(\partial_t-\partial_r)^2 Z^{\le 2} w\|_{L^2L^2(\tilde{C}^R_\tau)}\\ &+\|(\partial_t+\partial_r)^2 Z^{\le 2} w\|_{L^2L^2(\tilde{C}^R_\tau)}\\ &+\|(\partial_t^2-\partial_r^2) Z^{\le 2} w\|_{L^2L^2(\tilde{C}^R_\tau)}.\end{aligned} \tag{3.12}$$

Using this in (3.8) and the estimating via (3.9), (3.10), and (3.11) gives

$$\|\partial w\|_{L^\infty L^\infty(C^R_\tau)} \lesssim \frac{1}{\tau^{\frac{1}{2}} R^{\frac{3}{2}}}\|\partial Z^{\le 3} w\|_{L^2L^2(\tilde{C}^R_\tau)} + \frac{1}{\tau^{\frac{1}{2}} R^{\frac{1}{2}}}\|(\partial_t^2-\partial_r^2) Z^{\le 2} w\|_{L^2L^2(\tilde{C}^R_\tau)}.$$

Relying upon

$$\partial_t^2 - \partial_r^2 = \Box + \frac{2}{r}\partial_r + \nabla\!\!\!/ \cdot \nabla\!\!\!/ \tag{3.13}$$

and (1.5) yields (3.6).

For (3.7), we argue similarly. By (1.4) and (1.5) (applied to $\partial_r Z^{\le 2} w$), we obtain

$$\|\partial_r \nabla\!\!\!/ Z^{\le 2} w\|_{L^2L^2(\tilde{C}^U_\tau)} \lesssim \frac{1}{\tau}\|\partial Z^{\le 3} w\|_{L^2L^2(\tilde{C}^U_\tau)}.$$

And using (3.5) with w replaced by $(\partial_t-\partial_r)Z^{\le 2}w$ and $(\partial_r+\partial_r)Z^{\le 2}w$ respectively, we see that

$$\|(\partial_t-\partial_r)^2 Z^{\le 2} w\|_{L^2L^2(\tilde{C}^U_\tau)} \lesssim \frac{1}{U}\|\partial Z^{\le 3} w\|_{L^2L^2(\tilde{C}^U_\tau)} + \frac{\tau}{U}\|(\partial_t^2-\partial_r^2) Z^{\le 2} w\|_{L^2L^2(\tilde{C}^U_\tau)},$$

$$\|(\partial_t+\partial_r)^2 Z^{\le 2} w\|_{L^2L^2(\tilde{C}^U_\tau)} \lesssim \frac{1}{\tau}\|\partial Z^{\le 3} w\|_{L^2L^2(\tilde{C}^U_\tau)} + \|(\partial_t^2-\partial_r^2) Z^{\le 2} w\|_{L^2L^2(\tilde{C}^U_\tau)}.$$

Using these in (3.3) and the $\tilde{C}^U_\tau$ analog of (3.12), in combination with (3.13) and (1.5) as above, we see that

$$\|\partial w\|_{L^\infty L^\infty(C^U_\tau)} \lesssim \frac{1}{U^{\frac{1}{2}}\tau^{\frac{3}{2}}}\|\partial Z^{\le 2} w\|_{L^2L^2(\tilde{C}^U_\tau)} + \frac{U^{\frac{1}{2}}}{\tau^{\frac{3}{2}}}\|\partial_r \partial Z^{\le 2} w\|_{L^2L^2(\tilde{C}^U_\tau)}$$

$$\lesssim \frac{1}{U^{\frac{1}{2}}\tau^{\frac{3}{2}}}\|\partial Z^{\le 3} w\|_{L^2L^2(\tilde{C}_\tau^U)} + \frac{1}{U^{\frac{1}{2}}\tau^{\frac{1}{2}}}\|\Box Z^{\le 2} w\|_{L^2L^2(\tilde{C}_\tau^U)},$$

which completes the proof. □

4 Global existence

Here we provide the proof of Theorem 1.1. To do so, we set $u_0 \equiv 0$, $v_0 \equiv 0$ and recursively define u_k, v_k to solve

$$\begin{cases} \Box u_k = (\partial_t + \partial_r)u_{k-1}\,\partial_t v_{k-1} - \partial_r u_{k-1}(\partial_t + \partial_r)v_{k-1} - \not\nabla u_{k-1}\cdot \not\nabla v_{k-1}, \\ \Box v_k = \partial_t u_{k-1}\,\partial_t v_{k-1}, \\ (u_k(0,\,\cdot), \partial_t u_k(0,\,\cdot)) = (u_{(0)}, u_{(1)}), \\ (v_k(0,\,\cdot), \partial_t v_k(0,\,\cdot)) = (v_{(0)}, v_{(1)}). \end{cases} \tag{4.1}$$

We will show that the sequences (u_k) and (v_k) converge. The limits yield the desired solutions u, v to (1.1).

4.1 Boundedness

We fix $0 < p < 1$, $0 < \delta < \min\Big(p, 1-p\Big)$, and N large enough so that $\frac{N}{2} + 3 \le N$. We then set

$$\begin{aligned} M_k = {} & \|\langle r\rangle^{\frac{p-1}{2}} \not\partial Z^{\le N} u_k\|_{L^2L^2} + \|\langle r\rangle^{\frac{p-1}{2}} r^{-1} Z^{\le N} u_k\|_{L^2L^2} + \|\langle r\rangle^{\frac{p-1}{2}} \not\partial Z^{\le N} v_k\|_{L^2L^2} \\ & + \|\langle r\rangle^{\frac{p-1}{2}} r^{-1} Z^{\le N} v_k\|_{L^2L^2} + \|Z^{\le N} u_k\|_{LE^1} + \|\partial Z^{\le N} u_k\|_{L^\infty L^2} \\ & + \|\langle r\rangle^{-\frac{1+\delta}{2}} \partial Z^{\le N} v_k\|_{L^2L^2} + \|\langle r\rangle^{-\frac{\delta}{2}} \partial Z^{\le N} v_k\|_{L^\infty L^2} \\ & + \sup_\tau \sup_{R\le \tau/2} \Big(\tau^{\frac{1}{2}} R \|\partial Z^{\le \frac{N}{2}} u_k\|_{L^\infty L^\infty(C_\tau^R)}\Big) \\ & + \left[\sum_\tau \sum_{R\le\tau/2} \Big(\tau^{\frac{1}{2}} R^{1-\frac{\delta}{2}} \|\partial Z^{\le \frac{N}{2}} v_k\|_{L^\infty L^\infty(C_\tau^R)}\Big)^2\right]^{\frac{1}{2}} \\ & + \sup_\tau \sup_{U\le \tau/4} \Big(\tau U^{\frac{1}{2}} \|\partial Z^{\le \frac{N}{2}} u_k\|_{L^\infty L^\infty(C_\tau^U)}\Big) \\ & + \left[\sum_\tau \sum_{U\le\tau/4} \Big(\tau^{1-\frac{\delta}{2}} U^{\frac{1}{2}} \|\partial Z^{\le \frac{N}{2}} v_k\|_{L^\infty L^\infty(C_\tau^U)}\Big)^2\right]^{\frac{1}{2}}. \end{aligned} \tag{4.2}$$

For any $k \ge 1$, we shall show that

$$M_k \le C_0\varepsilon + C M_{k-1}^2 \tag{4.3}$$

for some fixed constant C_0. Provided that $\varepsilon > 0$ is sufficiently small, a straightforward induction argument then shows that

$$M_k \le 2C_0\varepsilon \tag{4.4}$$

for any k.

The product rule gives

$$|Z^{\le N}\Box u_k| \lesssim |\partial Z^{\le \frac{N}{2}} u_{k-1}||\partial Z^{\le N} v_{k-1}| + |\partial Z^{\le N} u_{k-1}||\partial Z^{\le \frac{N}{2}} v_{k-1}|. \tag{4.5}$$

Hence,

$$\begin{aligned}
&\|\langle r\rangle^{\frac{p+1}{2}}\Box Z^{\le N} u_k\|_{L^2L^2(C_\tau^R)}\\
&\quad\lesssim \tau^{-\frac{1}{2}}R^{\frac{p+\delta}{2}}\left(\tau^{\frac{1}{2}}R\|\partial Z^{\le\frac{N}{2}}u_{k-1}\|_{L^\infty L^\infty(C_\tau^R)}\right)\|\langle r\rangle^{-\frac{1+\delta}{2}}\partial Z^{\le N}v_{k-1}\|_{L^2L^2(C_\tau^R)}\\
&\qquad+\tau^{-\frac{1}{2}}R^{\frac{p+\delta}{2}}\left(\tau^{\frac{1}{2}}R^{1-\frac{\delta}{2}}\|\partial Z^{\le\frac{N}{2}}v_{k-1}\|_{L^\infty L^\infty(C_\tau^R)}\right)\|Z^{\le N}u_{k-1}\|_{LE^1},
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
&U^{\frac{1}{2}}\|\langle r\rangle^{\frac{p}{2}}\Box Z^{\le N} u_k\|_{L^2L^2(C_\tau^U)}\\
&\quad\lesssim \tau^{\frac{p-1+\delta}{2}}\left(U^{\frac{1}{2}}\tau\|\partial Z^{\le\frac{N}{2}}u_{k-1}\|_{L^\infty L^\infty(C_\tau^U)}\right)\|\langle r\rangle^{-\frac{1+\delta}{2}}\partial Z^{\le N}v_{k-1}\|_{L^2L^2(C_\tau^U)}\\
&\qquad+\tau^{\frac{p-1+\delta}{2}}\left(U^{\frac{1}{2}}\tau^{1-\frac{\delta}{2}}\|\partial Z^{\le\frac{N}{2}}v_{k-1}\|_{L^\infty L^\infty(C_\tau^U)}\right)\|Z^{\le N}u_{k-1}\|_{LE^1}.
\end{aligned} \tag{4.7}$$

From this, it follows that

$$\begin{aligned}
&\left(\sum_\tau\sum_{R\le\tau/2}\|\langle r\rangle^{\frac{1+p}{2}}\Box Z^{\le N}u_k\|^2_{L^2L^2(C_\tau^R)}\right)^{\frac{1}{2}}\\
&\qquad+\sum_U\left(\sum_{\tau\ge 4U}U\|\langle r\rangle^{\frac{p}{2}}\Box Z^{\le N}u_k\|^2_{L^2L^2(C_\tau^U)}\right)^{\frac{1}{2}}\lesssim M_{k-1}^2.
\end{aligned} \tag{4.8}$$

By (2.5) and (1.2), along with a Hardy inequality, we get

$$\|\langle r\rangle^{\frac{p-1}{2}}\bar\partial Z^{\le N}u_k\|_{L^2L^2} + \|\langle r\rangle^{\frac{p-1}{2}}r^{-1}Z^{\le N}u_k\|_{L^2L^2} \lesssim \varepsilon + M_{k-1}^2. \tag{4.9}$$

As the above argument does not rely on the null structure of $\Box u_k$ and we also have

$$|Z^{\le N}\Box v_k| \lesssim |\partial Z^{\le \frac{N}{2}} u_{k-1}||\partial Z^{\le N} v_{k-1}| + |\partial Z^{\le N} u_{k-1}||\partial Z^{\le \frac{N}{2}} v_{k-1}|, \tag{4.10}$$

the same arguments show that

$$\left(\sum_{\tau}\sum_{R\le\tau/2}\|\langle r\rangle^{\frac{1+p}{2}}\Box Z^{\le N}v_k\|^2_{L^2L^2(C^R_\tau)}\right)^{\frac12} + \sum_U\left(\sum_{\tau\ge 4U} U\|\langle r\rangle^{\frac{p}{2}}\Box Z^{\le N}v_k\|^2_{L^2L^2(C^U_\tau)}\right)^{\frac12}\lesssim M^2_{k-1}, \tag{4.11}$$

which we shall use later.

It is in the process of bounding $\|Z^{\le N}u_k\|_{LE^1}$ and $\|\partial Z^{\le N}u_k\|_{L^\infty L^2}$ that we will need the null condition. A finer alternative to (4.5) that takes care with the good directions is

$$|\Box Z^{\le N}u_k|\lesssim |Z^{\le\frac{N}{2}}\partial v_{k-1}||Z^{\le N}\bar\partial u_{k-1}| + |Z^{\le\frac{N}{2}}\bar\partial u_{k-1}||Z^{\le N}\partial v_{k-1}| + |Z^{\le\frac{N}{2}}\partial u_{k-1}||Z^{\le N}\bar\partial v_{k-1}| + |Z^{\le\frac{N}{2}}\bar\partial v_{k-1}||Z^{\le N}\partial u_{k-1}|. \tag{4.12}$$

We need to consider

$$\int_0^\infty \|Z^{\le N}\Box u_k(s,\,\cdot)\|_{L^2}\,ds \le \sum_{j\ge 0}\int_0^\infty \|Z^{\le N}\Box u_k(s,\,\cdot)\|_{L^2(A_{2^j})}\,ds.$$

To each lower order term in (4.12), we apply (3.1) to see that this is

$$\begin{aligned}\lesssim &\sum_{j\ge0}\int_0^\infty 2^{-\frac{p-\delta}{2}j}\|\langle r\rangle^{\frac{p-1}{2}}Z^{\le N}\bar\partial u_{k-1}(s,\,\cdot)\|_{L^2(A_{2^j})}\\ &\|\langle r\rangle^{-\frac{1+\delta}{2}}Z^{\le\frac{N}{2}+2}\partial v_{k-1}(s,\,\cdot)\|_{L^2(\tilde A_{2^j})}\,ds\\ &+\sum_{j\ge0}\int_0^\infty 2^{-\frac{p-\delta}{2}j}\|\langle r\rangle^{\frac{p-1}{2}}Z^{\le\frac{N}{2}+2}\bar\partial u_{k-1}(s,\,\cdot)\\ &\|_{L^2(\tilde A_{2^j})}\|\langle r\rangle^{-\frac{1+\delta}{2}}Z^{\le N}\partial v_{k-1}(s,\,\cdot)\|_{L^2(A_{2^j})}\,ds\\ &+\sum_{j\ge0}\int_0^\infty 2^{-\frac{p}{2}j}\|\langle r\rangle^{-\frac12}Z^{\le\frac{N}{2}+2}\partial u_{k-1}(s,\,\cdot)\\ &\|_{L^2(\tilde A_{2^j})}\|\langle r\rangle^{\frac{p-1}{2}}Z^{\le N}\bar\partial v_{k-1}(s,\,\cdot)\|_{L^2(A_{2^j})}\,ds\\ &+\sum_{j\ge0}\int_0^\infty 2^{-\frac{p}{2}j}\|\langle r\rangle^{-\frac12}Z^{\le N}\partial u_{k-1}(s,\,\cdot)\\ &\|_{L^2(A_{2^j})}\|\langle r\rangle^{\frac{p-1}{2}}Z^{\le\frac{N}{2}+2}\bar\partial v_{k-1}(s,\,\cdot)\|_{L^2(\tilde A_{2^j})}\,ds.\end{aligned}$$

By the Schwarz inequality and (1.3), this is

$$\lesssim \Big(\|\langle r\rangle^{\frac{p-1}{2}}\bar\partial Z^{\le N}u_{k-1}\|_{L^2L^2}+\|\langle r\rangle^{\frac{p-1}{2}}r^{-1}Z^{\le N}u_{k-1}\|_{L^2L^2}\Big)$$
$$\times\|\langle r\rangle^{-\frac{1+\delta}{2}}\partial Z^{\le N}v_{k-1}\|_{L^2L^2}$$
$$+\|Z^{\le N}u_{k-1}\|_{LE^1}\Big(\|\langle r\rangle^{\frac{p-1}{2}}\bar\partial Z^{\le N}v_{k-1}\|_{L^2L^2}+\|\langle r\rangle^{\frac{p-1}{2}}r^{-1}Z^{\le N}v_{k-1}\|_{L^2L^2}\Big)$$

provided $\frac{N}{2}+2\le N$, which is clearly $\mathcal{O}(M_{k-1}^2)$ as desired. Hence, due to (2.3) and (1.2), we have shown

$$\|Z^{\le N}u_k\|_{LE^1}+\|\partial Z^{\le N}u_k\|_{L^\infty L^2}\lesssim \varepsilon+M_{k-1}^2. \tag{4.13}$$

In order to address the v_k terms, we next consider

$$\|\langle r\rangle^{\frac{1-\delta}{2}}Z^{\le N}\Box v_k\|_{L^2L^2}.$$

To the lower order factors in (4.10) we apply (3.1) to see that this is

$$\lesssim \|\partial Z^{\le \frac{N}{2}+2}u_{k-1}\|_{L^\infty L^2}\|\langle r\rangle^{-\frac{1+\delta}{2}}\partial Z^{\le N}v_{k-1}\|_{L^2L^2}$$
$$+\|\partial Z^{\le N}u_{k-1}\|_{L^\infty L^2}\|\langle r\rangle^{-\frac{1+\delta}{2}}\partial Z^{\le \frac{N}{2}+2}v_{k-1}\|_{L^2L^2}.$$

Since $\frac{N}{2}+2\le N$, this is $\mathcal{O}(M_{k-1}^2)$. When combined with (1.2), (2.9), (4.11), and the observation that

$$\|\langle r\rangle^{-\frac{\delta}{2}}\partial Z^{\le N}v_k\|_{L^\infty L^2}\lesssim \|\langle r\rangle^{-\frac{\delta}{2}}(\partial_t-\partial_r)Z^{\le N}v_k\|_{L^\infty L^2}+\|\langle r\rangle^{\frac{p}{2}}\bar\partial Z^{\le N}v_k\|_{L^\infty L^2},$$

this gives

$$\|\langle r\rangle^{\frac{p-1}{2}}\bar\partial Z^{\le N}v_k\|_{L^2L^2}+\|\langle r\rangle^{\frac{p-1}{2}}r^{-1}Z^{\le N}v_k\|_{L^2L^2}+\|\langle r\rangle^{-\frac{1+\delta}{2}}\partial Z^{\le N}v_k\|_{L^2L^2}$$
$$+\|\langle r\rangle^{-\frac{\delta}{2}}\partial Z^{\le N}v_k\|_{L^\infty L^2}\lesssim \varepsilon+M_{k-1}^2. \tag{4.14}$$

In order to show (4.3), it remains to bound the $L^\infty L^\infty$ terms in (4.2). Applying (3.1) to each lower order piece in (4.12), we see that

$$R^{\frac12}\|Z^{\le N}\Box u_k\|_{L^2L^2(\tilde C_\tau^R)}\lesssim R^{\frac{\delta-p}{2}}\|\langle r\rangle^{-\frac{\delta}{2}}Z^{\le \frac{N}{2}+2}\partial v_{k-1}\|_{L^\infty L^2}\|\langle r\rangle^{\frac{p-1}{2}}Z^{\le N}\bar\partial u_{k-1}\|_{L^2L^2}$$
$$+R^{\frac{\delta-p}{2}}\|\langle r\rangle^{\frac{p-1}{2}}Z^{\le \frac{N}{2}+2}\bar\partial u_{k-1}\|_{L^2L^2}\|\langle r\rangle^{-\frac{\delta}{2}}Z^{\le N}\partial v_{k-1}\|_{L^\infty L^2}$$
$$+R^{-\frac{p}{2}}\|Z^{\le \frac{N}{2}+2}\partial u_{k-1}\|_{L^\infty L^2}\|\langle r\rangle^{\frac{p-1}{2}}Z^{\le N}\bar\partial v_{k-1}\|_{L^2L^2}$$
$$+R^{-\frac{p}{2}}\|Z^{\le N}\partial u_{k-1}\|_{L^\infty L^2}\|\langle r\rangle^{\frac{p-1}{2}}Z^{\le \frac{N}{2}+2}\bar\partial v_{k-1}\|_{L^2L^2}.$$

And thus, by (3.6) and the facts that $\frac{N}{2}+2\le N$ and $0<\delta<p$,

$$\begin{aligned}\tau^{\frac{1}{2}} R \|\partial Z^{\le \frac{N}{2}} u_k\|_{L^\infty L^\infty(C_\tau^R)} &\lesssim \|\langle r\rangle^{-\frac{1}{2}} \partial Z^{\le \frac{N}{2}+3} u_k\|_{L^2L^2(\tilde{C}_\tau^R)} \\ &+ \|\langle r\rangle^{-\frac{\delta}{2}} Z^{\le N} \partial v_{k-1}\|_{L^\infty L^2} \|\langle r\rangle^{\frac{p-1}{2}} Z^{\le N} \bar{\partial} u_{k-1}\|_{L^2L^2} \\ &+ \|Z^{\le N} \partial u_{k-1}\|_{L^\infty L^2} \|\langle r\rangle^{\frac{p-1}{2}} Z^{\le N} \bar{\partial} v_{k-1}\|_{L^2L^2}\end{aligned}$$

for $1 \le R \le \tau/2$. When this is combined with (1.3) and (4.13) it yields that

$$\sup_\tau \sup_{R\le \tau/2} \Big(\tau^{\frac{1}{2}} R \|\partial Z^{\le \frac{N}{2}} u_k\|_{L^\infty L^\infty(C_\tau^R)}\Big) \lesssim \varepsilon + M_{k-1}^2. \tag{4.15}$$

Similarly using (3.1) and (1.3) in (4.10) instead gives

$$\begin{aligned}\|Z^{\le N} \Box v_k\|_{L^2L^2(\tilde{C}_\tau^R)} &\lesssim R^{-\frac{1-\delta}{2}} \|\partial Z^{\le \frac{N}{2}+2} u_{k-1}\|_{L^\infty L^2} \|\langle r\rangle^{-\frac{1+\delta}{2}} \partial Z^{\le N} v_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} \\ &+ R^{-\frac{1-\delta}{2}} \|\langle r\rangle^{-\frac{1+\delta}{2}} \partial Z^{\le \frac{N}{2}+2} v_{k-1}\|_{L^2L^2(\tilde{\tilde{C}}_\tau^R)} \|\partial Z^{\le N} u_{k-1}\|_{L^\infty L^2}.\end{aligned}$$

When combined with (3.6), this yields

$$\begin{aligned}\tau^{\frac{1}{2}} R^{1-\frac{\delta}{2}} \|\partial Z^{\le \frac{N}{2}} v_k\|_{L^\infty L^\infty(C_\tau^R)} &\lesssim \|\langle r\rangle^{-\frac{1+\delta}{2}} Z^{\le \frac{N}{2}+3} v_k\|_{L^2L^2(\tilde{C}_\tau^R)} \\ &+ \|\partial Z^{\le \frac{N}{2}+2} u_{k-1}\|_{L^\infty L^2} \|\langle r\rangle^{-\frac{1+\delta}{2}} \partial Z^{\le N} v_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} \\ &+ \|\langle r\rangle^{-\frac{1+\delta}{2}} \partial Z^{\le \frac{N}{2}+2} v_{k-1}\|_{L^2L^2(\tilde{\tilde{C}}_\tau^R)} \|\partial Z^{\le N} u_{k-1}\|_{L^\infty L^2},\end{aligned}$$

which upon pairing with (4.14) gives

$$\left[\sum_\tau \sum_{R\le \tau/4} \Big(R^{\frac{3-\delta}{2}} \|\partial Z^{\le \frac{N}{2}} v_k\|_{L^\infty L^\infty(C_\tau^R)}\Big)^2\right]^{\frac{1}{2}} \lesssim \varepsilon + M_{k-1}^2. \tag{4.16}$$

Using (3.7) in place of (3.6), these same arguments show

$$\begin{aligned}U^{\frac{1}{2}} \tau \|\partial Z^{\le \frac{N}{2}} u_k\|_{L^\infty L^\infty(C_\tau^U)} &\lesssim \|\langle r\rangle^{-\frac{1}{2}} Z^{\le \frac{N}{2}+3} u_k\|_{L^2L^2(\tilde{C}_\tau^U)} \\ &+ \|\langle r\rangle^{-\frac{\delta}{2}} Z^{\le N} \partial v_{k-1}\|_{L^\infty L^2} \|\langle r\rangle^{\frac{p-1}{2}} Z^{\le N} \bar{\partial} u_{k-1}\|_{L^2L^2} \\ &+ \|Z^{\le N} \partial u_{k-1}\|_{L^\infty L^2} \|\langle r\rangle^{\frac{p-1}{2}} Z^{\le N} \bar{\partial} v_{k-1}\|_{L^2L^2},\end{aligned}$$

and

$$\begin{aligned}U^{\frac{1}{2}} \tau^{1-\frac{\delta}{2}} \|\partial Z^{\le \frac{N}{2}} v_k\|_{L^\infty L^\infty(C_\tau^U)} &\lesssim \|\langle r\rangle^{-\frac{1+\delta}{2}} Z^{\le \frac{N}{2}+3} v_k\|_{L^2L^2(\tilde{C}_\tau^U)} \\ &+ \|\partial Z^{\frac{N}{2}+2} u_{k-1}\|_{L^\infty L^2} \|\langle r\rangle^{-\frac{1+\delta}{2}} \partial Z^{\le N} v_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} \\ &+ \|\langle r\rangle^{-\frac{1+\delta}{2}} \partial Z^{\le \frac{N}{2}+2} v_{k-1}\|_{L^2L^2(\tilde{\tilde{C}}_\tau^U)} \|\partial Z^{\le N} u_{k-1}\|_{L^\infty L^2}.\end{aligned}$$

When these are combined with (4.13) and (4.14) respectively, we obtain

$$\sup_{\tau}\sup_{U\le\tau/4}\left(\tau U^{\frac{1}{2}}\|\partial Z^{\le\frac{N}{2}}u_k\|_{L^\infty L^\infty(C_\tau^U)}\right) + \left[\sum_{\tau}\sum_{U\le\tau/4}\left(\tau^{1-\frac{\delta}{2}}U^{\frac{1}{2}}\|\partial Z^{\le\frac{N}{2}}v_k\|_{L^\infty L^\infty(C_\tau^U)}\right)^2\right]^{\frac{1}{2}} \lesssim \varepsilon + M_{k-1}^2. \quad (4.17)$$

The combination of (4.9), (4.13), (4.14), (4.15), (4.16), and (4.17) prove (4.3) and, hence, (4.4) as desired.

4.2 Convergence

It remains to show that the sequence (u_k) and (v_k) converge. We set

$$\begin{aligned}
A_k = {} & \|\langle r\rangle^{\frac{p-1}{2}}\not\partial Z^{\le N}(u_k-u_{k-1})\|_{L^2L^2} + \|\langle r\rangle^{\frac{p-1}{2}}r^{-1}Z^{\le N}(u_k-u_{k-1})\|_{L^2L^2}\\
&+\|\langle r\rangle^{\frac{p-1}{2}}\not\partial Z^{\le N}(v_k-v_{k-1})\|_{L^2L^2}\\
&+\|\langle r\rangle^{\frac{p-1}{2}}r^{-1}Z^{\le N}(v_k-v_{k-1})\|_{L^2L^2} + \|Z^{\le N}(u_k-u_{k-1})\|_{LE^1}\\
&+\|\partial Z^{\le N}(u_k-u_{k-1})\|_{L^\infty L^2}\\
&+\|\langle r\rangle^{-\frac{1+\delta}{2}}\partial Z^{\le N}(v_k-v_{k-1})\|_{L^2L^2}\\
&+\sup_{\tau}\sup_{R\le\tau/2}\left(\tau^{\frac{1}{2}}R\|\partial Z^{\le\frac{N}{2}}(u_k-u_{k-1})\|_{L^\infty L^\infty(C_\tau^R)}\right)\\
&+\left[\sum_{\tau}\sum_{R\le\tau/2}\left(R^{\frac{3-\delta}{2}}\|\partial Z^{\le\frac{N}{2}}(v_k-v_{k-1})\|_{L^\infty L^\infty(C_\tau^R)}\right)^2\right]^{\frac{1}{2}}\\
&+\sup_{\tau}\sup_{U\le\tau/4}\left(\tau U^{\frac{1}{2}}\|\partial Z^{\le\frac{N}{2}}(u_k-u_{k-1})\|_{L^\infty L^\infty(C_\tau^U)}\right)\\
&+\left[\sum_{\tau}\sum_{U\le\tau/4}\left(\tau^{1-\frac{\delta}{2}}U^{\frac{1}{2}}\|\partial Z^{\le\frac{N}{2}}(v_k-v_{k-1})\|_{L^\infty L^\infty(C_\tau^U)}\right)^2\right]^{\frac{1}{2}}.
\end{aligned} \quad (4.18)$$

We seek to show that

$$A_k \le \frac{1}{2}A_{k-1}, \quad (4.19)$$

which implies that the sequences are Cauchy and thus convergent.

We note that

$$\begin{aligned}|Z^{\le N}\Box(u_k - u_{k-1})| &\lesssim |\partial Z^{\le \frac{N}{2}}(u_{k-1}-u_{k-2})||\partial Z^{\le N} v_{k-1}| \\ &+|\partial Z^{\le N}(u_{k-1}-u_{k-2})||\partial Z^{\le \frac{N}{2}} v_{k-1}| \\ &+|\partial Z^{\le \frac{N}{2}} u_{k-2}||\partial Z^{\le N}(v_{k-1}-v_{k-2})| + |\partial Z^{\le N} u_{k-2}||\partial Z^{\le \frac{N}{2}}(v_{k-1}-v_{k-2})|,\end{aligned} \tag{4.20}$$

$$\begin{aligned}|Z^{\le N}\Box(v_k - v_{k-1})| &\lesssim |\partial Z^{\le \frac{N}{2}}(u_{k-1}-u_{k-2})||\partial Z^{\le N} v_{k-1}| \\ &+|\partial Z^{\le N}(u_{k-1}-u_{k-2})||\partial Z^{\le \frac{N}{2}} v_{k-1}| \\ &+|\partial Z^{\le \frac{N}{2}} u_{k-2}||\partial Z^{\le N}(v_{k-1}-v_{k-2})| + |\partial Z^{\le N} u_{k-2}||\partial Z^{\le \frac{N}{2}}(v_{k-1}-v_{k-2})|,\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}|\Box Z^{\le N}(u_k - u_{k-1})| &\lesssim |Z^{\le \frac{N}{2}}\partial v_{k-1}||Z^{\le N}\bar\partial(u_{k-1}-u_{k-2})| \\ &+|Z^{\le N}\partial v_{k-1}||Z^{\le \frac{N}{2}}\bar\partial(u_{k-1}-u_{k-2})| \\ &+|Z^{\le \frac{N}{2}}\bar\partial u_{k-2}||Z^{\le N}\partial(v_{k-1}-v_{k-2})| + |Z^{\le N}\bar\partial u_{k-2}||Z^{\le \frac{N}{2}}\partial(v_{k-1}-v_{k-2})| \\ &+|Z^{\le \frac{N}{2}}\bar\partial v_{k-1}||Z^{\le N}\partial(u_{k-1}-u_{k-2})| + |Z^{\le N}\bar\partial v_{k-1}||Z^{\le \frac{N}{2}}\partial(u_{k-1}-u_{k-2})| \\ &+|Z^{\le \frac{N}{2}}\bar\partial(v_{k-1}-v_{k-2})||Z^{\le N}\partial u_{k-2}| + |Z^{\le N}\bar\partial(v_{k-1}-v_{k-2})||Z^{\le \frac{N}{2}}\partial u_{k-2}|,\end{aligned} \tag{4.22}$$

which will be used in place of (4.5), (4.10), and (4.12) respectively. Arguing as in the proof of (4.3) then shows that

$$A_k \lesssim (M_{k-1} + M_{k-2})A_{k-1}.$$

Provided that ε is sufficiently small, an application of (4.4) immediately yields (4.19) and completes the proof.

We end with a brief remark about the asymptotics of the solution. The solution u is also bounded in the norms given by (4.2). Indeed, by examining the last two terms, one can immediately observe that u as more rapid asymptotic decay $\mathcal{O}(t^{-1})$ than the component v, which instead is $\mathcal{O}(t^{-1+\frac{\delta}{2}})$.

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Declarations

Conflict of interest The author(s) declare that they have no competing interests

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