A LOCAL ENERGY ESTIMATE FOR 2-DIMENSIONAL DIRICHLET WAVE EQUATIONS

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Abstract. We examine a variant of the integrated local energy estimate for $(1+2)$-dimensional Dirichlet wave equations exterior to star-shaped obstacles. The classical bound on the solution, rather than the derivative, is not typically available in two spatial dimensions. Using an argument inspired by the r^p-weighted method of Dafermos and Rodnianski and taking advantange of the Dirichlet boundary conditions allow for the recovery of such a term when the initial energy is appropriately weighted.

1. Introduction

We develop a variant of the integrated local energy estimate that holds for two-dimensional wave equations exterior to star-shaped obstacles with Dirichlet boundary conditions. Integrated local energy estimates first appeared in [12] and are known to hold for wave equations with spatial dimension $n \geq 3$. The same are known to hold for Dirichlet wave equations exterior to star-shaped obstacles as the boundary terms that arise upon integrating by parts have a favorable sign. These, now standard, arguments are known to fail in two spatial dimensions. We introduce a novel variant in two dimensions that recovers portions of the integrated local energy bound.

For $\Box = \partial_t^2 - \Delta$ where $\Delta u = \nabla \cdot \nabla u = \sum_{i=1}^n \partial_{x_i}^2 u$, we shall examine the initial / boundary value problem

$$\begin{cases} \Box u = 0, \quad (t,\mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^n \backslash \mathcal{K}, \\ u(t,\mathbf{x}) = 0, \quad \forall\, \mathbf{x} \in \partial\mathcal{K} \text{ and } t \geq 0, \\ u(0,\,\cdot\,) = u_0, \quad \partial_t u(0,\,\cdot\,) = u_1. \end{cases} \tag{1.1}$$

Here $\mathcal{K} \neq \emptyset$ is an open, bounded, star-shaped set with smooth boundary. By translation symmetry, we may assume without loss of generality that $0 \in \mathcal{K}$ and that $\mathcal{K}$ is star-shaped with respect to the origin. In this case, if $\mathbf{n}$ is the outward pointing normal to $\mathcal{K}$ at any point $\mathbf{x} \in \partial\mathcal{K}$, then

$$\mathbf{x} \cdot \mathbf{n} \geq 0. \tag{1.2}$$

By scaling, we may assume without loss of generality that $\{|\mathbf{x}| \leq e^2\} \subset \mathcal{K}$.

If we allow $\partial u = (\partial_t u, \nabla_x u)$, the integrated local energy estimate, which is known to hold for $n \geq 3$, states that solutions to (1.1) satisfy

$$\int |\partial u(T,\mathbf{x})|^2\,d\mathbf{x} + R^{-1}\int_0^T \int_{|\mathbf{x}|\leq R} |\partial u(t,\mathbf{x})|^2\,d\mathbf{x}\,dt + R^{-3}\int_0^T\int_{|\mathbf{x}|\leq R} |u(t,\mathbf{x})|^2\,d\mathbf{x}\,dt \lesssim \int |\partial u(0,\mathbf{x})|^2\,d\mathbf{x}. \tag{1.3}$$

The implicit constant in the estimate is independent of the parameters R and T. The first term in the left is the conserved energy for $\Box$. The latter two terms capture the dispersive nature of the wave equation. The bound on these terms shows that the local energy (i.e. that within the compact set $\{|\mathbf{x}| \leq R\}$), when appropriately weighted to account for the size of the set, must decay sufficiently rapidly to be globally integrable. In dimensions $n \geq 4$, the last term in the left side may instead be replaced by

$$\int_0^T \int_{\mathbb{R}^n} \frac{1}{|\mathbf{x}|^3} |u(t,\mathbf{x})|^2\,d\mathbf{x}\,dt,$$

which is a slight improvement.

2020 *Mathematics Subject Classification.* 35L71, 35L05.

Key words and phrases. Wave equation, local energy estimate exterior domain.

The results contained herein were developed as a part of KH's Extended Essay for his International Baccalaureate. JM was supported in part by Simons Foundation Collaboration Grant 711724 and NSF grants DMS-2054910 and DMS-2135998.

Integrated local energy estimates have a rich history. They originated in the study of scattering theory. See, e.g., [12]. They have subsequently, in e.g. [6], [8], found applications in existence proofs for nonlinear wave equations. This includes playing a major role in the study of black hole stability [2], [7]. In the asymptotically flat regime, other common measures of dispersion such as Strichartz estimates ([4], [10]) and pointwise decay estimates ([15], [11], [3]) are known to be consequences of the integrated local energy estimate.

The estimate (1.3) is typically proved by pairing the equation $\Box u = 0$ with a multiplier of the form $\partial_t u + \frac{r}{r+R}\partial_r u + \frac{n-1}{2}\frac{1}{r+R}u$ where $R > 0$, and integrating by parts. Here $r = |\mathbf{x}|$ and $\partial_r = \frac{\mathbf{x}}{r}\cdot\nabla$. See [14] on $\mathbb{R}_+ \times \mathbb{R}^n$ and [8] exterior to star-shaped obstacles. See, e.g., [9] for generalizations and a more complete history.

In two spatial dimensions, the full boundaryless estimate (1.3) does not hold. The third term in the left poses the difficulty. Indeed consider

$$\begin{cases} \Box u(t,\mathbf{x}) = 0, \quad (t,\mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ u(0,\mathbf{x}) = \beta(|\mathbf{x}|/\rho), \quad \partial_t u(0,\mathbf{x}) = 0 \end{cases}$$

where β is a smooth, nonegative cutoff function with $\beta(r) \equiv 1$ for $r \le 1$ and $\beta(r) \equiv 0$ for $r \ge 2$. Due to the finite speed of propagation, $u(t,\mathbf{x}) \equiv 1$ for $t + |\mathbf{x}| \le \rho$. If (1.3) held, then there would be a fixed constant C so that

$$\pi(\rho - 1) = \int_0^{\rho-1}\int_{|\mathbf{x}|\le 1} |u(t,\mathbf{x})|^2\,d\mathbf{x}\,dt \le C\frac{1}{\rho^2}\int |\beta'(\mathbf{x}/\rho)|^2\,d\mathbf{x} = \tilde{C},$$

where $\tilde{C}$ is independent of ρ. For ρ sufficiently large, this produces a contradiction.

Within the typical proof of (1.3), the third term corresponds to the $\frac{n-1}{2}\frac{1}{r+R}u$ portion of the multiplier, which cancels out an unsigned occurrence of the Lagrangian that results from the other portion of the multiplier. The coefficient follows from a lower bound on

$$-\frac{n-1}{4}\Delta\Big(\frac{1}{r+R}\Big).$$

In two dimensions, this quantity is not beneficially signed.

Portions of (1.3) may be recovered in two dimensions. The first term corresponds to standard conservation of energy. And the bound for the second term roughly corresponds to, e.g., [13, Lemma 2.2] or to the $s = 1/2$ boundary of [4, (3.6)]. The bound on the third term of (1.3) instead corresponds to the $s = 3/2$ case of [4, (3.6)], which is out of reach when $n = 2$. The boundaryless case is particularly difficult due to low frequency contributions that frequently require moment conditions to recover local energy decay. See, e.g., [1], [5], [10], [16].

The main result of this paper is the following $(1 + 2)$-dimensional variant of the local energy estimate. It recovers a bound on the lower order term provided the energy contributions are sufficiently weighted.

Theorem 1.1. *Let $0 \in \mathcal{K} \subset \mathbb{R}^2$ be an open set with smooth boundary that is star-shaped with respect to the origin. Assume that*

$$\{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| \le e^2\} \subseteq \mathcal{K}.$$

Let $u \in C^2(\mathbb{R}_+ \times \mathbb{R}^2)$ be a solution to (1.1), and assume that for every $T > 0$ there is a $R > 0$ so that $u(t,x) = 0$ for $t \in [0,T]$ and $|\mathbf{x}| \ge R$. Then provided that $0 \le p \le 1$ and $T > 0$, we have

$$\begin{aligned} &\int_{\mathbb{R}^2\backslash\mathcal{K}} r(\ln r)^p\Big\{\Big[\Big(\partial_t + \partial_r + \frac{1}{2r}\Big)u(T,\mathbf{x})\Big]^2 + |\not\nabla u(T,\mathbf{x})|^2 + \frac{(u(T,\mathbf{x}))^2}{r^2}\Big\}\,d\mathbf{x} \\ &+ \int_0^T\int_{\mathbb{R}^2\backslash\mathcal{K}} (\ln r)^p\Big\{\Big[\Big(\partial_t + \partial_r\Big)u(t,\mathbf{x})\Big]^2 + |\not\nabla u(t,\mathbf{x})|^2\Big\}\,d\mathbf{x}\,dt + \int_0^T\int_{\mathbb{R}^2\backslash\mathcal{K}} \frac{p(1-p)}{r^2(\ln r)^{2-p}}(u(t,\mathbf{x}))^2\,d\mathbf{x}\,dt \\ &\lesssim \int_{\mathbb{R}^2\backslash\mathcal{K}} r(\ln r)^p\Big\{\Big[\Big(\partial_t + \partial_r + \frac{1}{2r}\Big)u(0,\mathbf{x})\Big]^2 + |\not\nabla u(0,\mathbf{x})|^2 + \frac{(u(0,\mathbf{x}))^2}{r^2}\Big\}\,d\mathbf{x}. \end{aligned} \tag{1.4}$$

Here the angular derivatives $\not\nabla$ are defined via the orthogonal decomposition

$$\nabla u = \frac{\mathbf{x}}{r}\partial_r u + \not\nabla u. \tag{1.5}$$

The implicit constant here is independent of T. Recall that the assumptions that $0 \in \mathcal{K}$ and $\{|\mathbf{x}| \leq e^2\} \subseteq \mathcal{K}$ can be made without loss of generality due to translation and scaling invariance respectively. Moreover, when the data are compactly supported, the condition that $u(t,\mathbf{x})$ vanishes for sufficiently large $|\mathbf{x}|$ is an immediate consequence of the finite speed of propagation. And for more general data, one can approximate by compactly supported data.

We note that (1.4) may be paired with the typical multiplier described above to partially recover (1.3). In this case, however, portions of the data will instead be measured in the weighted spaces that appear in the right side of (1.4). Perhaps of more significant consequence in applications is the corresponding weights that result on the forcing term when considering inhomogeneous equations.

The estimate (1.4) is most akin to the r^p-weighted estimates of [3] on $\mathbb{R}_+ \times \mathbb{R}^3$. It is understood that the components of ∂u that are tangent to the light cone decay more rapidly. In [3], multipliers of the form $r^p\Big(\partial_t + \partial_r + \frac{1}{r}\Big)$ with $0 < p < 2$ were used and an improvement over (1.3) was obtained for the good directions, though with a weighted initial energy. Our related strategy will use multipliers like

$$r(\ln r)^p\Big(\partial_t + \partial_r + \frac{1}{2r}\Big)$$

with $0 \leq p \leq 1$. The bound that is obtained only holds for the good directions, but it does yield control on the solution u itself in an appropriately weighted norm. With the exception of $p = 0$, this method relies heavily upon being in an exterior domain with Dirichlet boundary conditions.

2. Proof of Theorem 1.1

For a function $f(r)$ that will be fixed later, we consider

(2.1)

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}^2\backslash\mathcal{K}} \Box u(t,\mathbf{x})\, f(r)\Big(\partial_t + \partial_r + \frac{1}{2r}\Big)u(t,\mathbf{x})\, d\mathbf{x}\, dt \\ &= \int_0^T \int_{\mathbb{R}^2\backslash\mathcal{K}} (\partial_t^2 - \nabla\cdot\nabla)u(t,\mathbf{x})\, f(r)\partial_t u(t,\mathbf{x})\, d\mathbf{x}\, dt + \int_0^T \int_{\mathbb{R}^2\backslash\mathcal{K}} (\partial_t^2 - \nabla\cdot\nabla)u(t,\mathbf{x})\, f(r)\partial_r u(t,\mathbf{x})\, d\mathbf{x}\, dt \\ &\qquad + \int_0^T \int_{\mathbb{R}^2\backslash\mathcal{K}} (\partial_t^2 - \nabla\cdot\nabla)u(t,\mathbf{x})\, f(r)\frac{1}{2r}u(t,\mathbf{x})\, d\mathbf{x}\, dt. \end{aligned}$$

We will now manipulate each of the three integrals in the right side.

For the first, we use the chain rule and the Divergence Theorem (along with the Dirichlet boundary conditions) to compute

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^2\backslash\mathcal{K}} (\partial_t^2 - \nabla\cdot\nabla)u(t,\mathbf{x})\, f(r)\partial_t u(t,\mathbf{x})\, d\mathbf{x}\, dt &= \frac{1}{2}\int_0^T \int_{\mathbb{R}^2\backslash\mathcal{K}} f(r)\partial_t(\partial_t u(t,\mathbf{x}))^2\, d\mathbf{x}\, dt \\ + \int_0^T \int_{\mathbb{R}^2\backslash\mathcal{K}} f'(r)\partial_r u(t,\mathbf{x})\partial_t u(t,\mathbf{x})\, d\mathbf{x}\, dt &+ \frac{1}{2}\int_0^T \int_{\mathbb{R}^2\backslash\mathcal{K}} f(r)\partial_t|\nabla u(t,\mathbf{x})|^2\, d\mathbf{x}\, dt. \end{aligned} \tag{2.2}$$

Here we note that as $u(t,\mathbf{x}) = 0$ for all $\mathbf{x} \in \partial\mathcal{K}$ and $t \geq 0$, it follows that $\partial_t u(t,\mathbf{x}) = 0$ for all $\mathbf{x} \in \partial\mathcal{K}$ and $t \geq 0$. For later purposes we also note that

$$\mathbf{x} \in \partial\mathcal{K} \implies \nabla u(t,\mathbf{x}) = \mathbf{n}(\mathbf{x})\partial_{\mathbf{n}} u(t,\mathbf{x})$$

where $\mathbf{n}(\mathbf{x})$ denotes the outward unit normal to $\mathcal{K}$ at the point $\mathbf{x} \in \partial\mathcal{K}$ and $\partial_{\mathbf{n}}$ is the directional derivative in the direction $\mathbf{n}$. Using that $\Box u = 0$ and applying the Fundamental Theorem of Calculus with (2.2) then give that

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}} f(r)(\partial_t u(T,\mathbf{x}))^2\, d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}} f(r)|\nabla u(T,\mathbf{x})|^2\, d\mathbf{x} + \int_0^T \int_{\mathbb{R}^2\backslash\mathcal{K}} f'(r)\partial_r u(t,\mathbf{x})\partial_t u(t,\mathbf{x})\, d\mathbf{x}\, dt \\ = \frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}} f(r)(\partial_t u(0,\mathbf{x}))^2\, d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}} f(r)|\nabla u(0,\mathbf{x})|^2\, d\mathbf{x}. \end{aligned} \tag{2.3}$$

For the second integral in the right side of (2.1), integration by parts and the Divergence Theorem show that

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^2\setminus\mathcal{K}} (\partial_t^2 - \nabla\cdot\nabla)u(t,\mathbf{x})\, f(r)\partial_r u(t,\mathbf{x})\, d\mathbf{x}\, dt &= \int_{\mathbb{R}^2\setminus\mathcal{K}} f(r)\partial_t u(t,\mathbf{x})\partial_r u(t,\mathbf{x})\, dx\Big|_{t=0}^{T} \\
&\quad - \frac{1}{2}\int_0^T \int_{\mathbb{R}^2\setminus\mathcal{K}} f(r)\partial_r(\partial_t u(t,\mathbf{x}))^2\, d\mathbf{x}\, dt + \int_0^T \int_{\partial\mathcal{K}} \Big(\frac{\mathbf{x}}{r}\cdot\mathbf{n}(\mathbf{x})\Big) f(r)(\partial_{\mathbf{n}} u(t,\mathbf{x}))^2\, d\sigma(\mathbf{x})\, dt \\
&\quad + \int_0^T \int_{\mathbb{R}^2\setminus\mathcal{K}} f'(r)(\partial_r u(t,\mathbf{x}))^2\, d\mathbf{x}\, dt + \int_0^T \int_{\mathbb{R}^2\setminus\mathcal{K}} f(r)\nabla u(t,\mathbf{x})\cdot\nabla\partial_r u(t,\mathbf{x})\, d\mathbf{x}\, dt.
\end{aligned}$$

Using that

$$\nabla\partial_r u(t,\mathbf{x}) = \partial_r \nabla u(t,\mathbf{x}) + \frac{1}{r}\not\nabla u(t,\mathbf{x})$$

and the fact that the decomposition (1.5) is orthogonal, we then see that the right side is equal to

$$\begin{aligned}
&\int_{\mathbb{R}^2\setminus\mathcal{K}} f(r)\partial_t u(t,\mathbf{x})\partial_r u(t,\mathbf{x})\, d\mathbf{x}\Big|_{t=0}^{T} - \frac{1}{2}\int_0^T \int_{\mathbb{R}^2\setminus\mathcal{K}} f(r)\partial_r(\partial_t u(t,\mathbf{x}))^2\, d\mathbf{x}\, dt \\
&\quad + \int_0^T \int_{\partial\mathcal{K}} \Big(\frac{\mathbf{x}}{r}\cdot\mathbf{n}(\mathbf{x})\Big) f(r)(\partial_{\mathbf{n}} u(t,\mathbf{x}))^2\, d\sigma(\mathbf{x})\, dt + \int_0^T \int_{\mathbb{R}^2\setminus\mathcal{K}} f'(r)(\partial_r u(t,\mathbf{x}))^2\, d\mathbf{x}\, dt \\
&\quad + \int_0^T \int_{\mathbb{R}^2\setminus\mathcal{K}} \frac{f(r)}{r}|\not\nabla u(t,\mathbf{x})|^2\, d\mathbf{x}\, dt + \frac{1}{2}\int_0^T \int_{\mathbb{R}^2\setminus\mathcal{K}} f(r)\partial_r|\nabla u(t,\mathbf{x})|^2\, d\mathbf{x}\, dt.
\end{aligned}$$

The Divergence Theorem gives

$$\begin{aligned}
&- \frac{1}{2}\int_0^T \int_{\mathbb{R}^2\setminus\mathcal{K}} f(r)\partial_r\Big((\partial_t u(t,\mathbf{x}))^2 - |\nabla u(t,\mathbf{x})|^2\Big)\, d\mathbf{x}\, dt \\
&\quad = \frac{1}{2}\int_0^T \int_{\mathbb{R}^2\setminus\mathcal{K}} \nabla\cdot\Big(\frac{\mathbf{x}}{r}f(r)\Big)\Big((\partial_t u(t,\mathbf{x}))^2 - |\nabla u(t,\mathbf{x})|^2\Big)\, d\mathbf{x}\, dt \\
&\qquad - \frac{1}{2}\int_0^T \int_{\partial\mathcal{K}} \Big(\frac{\mathbf{x}}{r}\cdot\mathbf{n}(\mathbf{x})\Big) f(r)(\partial_{\mathbf{n}} u(t,\mathbf{x}))^2\, d\sigma(\mathbf{x})\, dt.
\end{aligned}$$

Since $\nabla\cdot\Big(\frac{\mathbf{x}}{r}f(r)\Big) = f'(r) + \frac{f(r)}{r}$ and since the orthogonality of (1.5) gives that $|\nabla u|^2 = (\partial_r u)^2 + |\not\nabla u|^2$, the assumption that $\Box u(t,\mathbf{x}) = 0$ then shows

$$\begin{aligned}
&\int_{\mathbb{R}^2\setminus\mathcal{K}} f(r)\partial_t u(T,\mathbf{x})\partial_r u(T,\mathbf{x})\, d\mathbf{x} + \frac{1}{2}\int_0^T \int_{\mathbb{R}^2\setminus\mathcal{K}} f'(r)(\partial_t u(t,\mathbf{x}))^2\, d\mathbf{x}\, dt \\
&\quad + \frac{1}{2}\int_0^T \int_{\mathbb{R}^2\setminus\mathcal{K}} f'(r)(\partial_r u(t,\mathbf{x}))^2\, d\mathbf{x}\, dt + \int_0^T \int_{\mathbb{R}^2\setminus\mathcal{K}} \Big(\frac{f(r)}{r} - \frac{1}{2}f'(r)\Big)|\not\nabla u(t,\mathbf{x})|^2\, d\mathbf{x}\, dt \\
&\quad + \frac{1}{2}\int_0^T \int_{\mathbb{R}^2\setminus\mathcal{K}} \frac{f(r)}{r}\Big((\partial_t u(t,\mathbf{x}))^2 - |\nabla u(t,\mathbf{x})|^2\Big)\, d\mathbf{x}\, dt + \frac{1}{2}\int_0^T \int_{\partial\mathcal{K}} \Big(\frac{\mathbf{x}}{r}\cdot\mathbf{n}(\mathbf{x})\Big) f(r)(\partial_{\mathbf{n}} u(t,\mathbf{x}))^2\, d\sigma(\mathbf{x})\, dt \\
&\quad = \int_{\mathbb{R}^2\setminus\mathcal{K}} f(r)\partial_t u(0,\mathbf{x})\partial_r u(0,\mathbf{x})\, d\mathbf{x}.
\end{aligned} \tag{2.4}$$

We finally consider the last integral in (2.1):

$$\begin{aligned}
\frac{1}{2}\int_0^T \int_{\mathbb{R}^2\setminus\mathcal{K}} (\partial_t^2 - \nabla\cdot\nabla)u(t,\mathbf{x})\, \frac{f(r)}{r}u(t,\mathbf{x})\, d\mathbf{x}\, dt &= \frac{1}{2}\int_{\mathbb{R}^2\setminus\mathcal{K}} \frac{f(r)}{r}u(t,\mathbf{x})\partial_t u(t,\mathbf{x})\, d\mathbf{x}\Big|_{t=0}^{T} \\
&\quad - \frac{1}{2}\int_0^T \int_{\mathbb{R}^2\setminus\mathcal{K}} \frac{f(r)}{r}\Big((\partial_t u(t,\mathbf{x}))^2 - |\nabla u(t,\mathbf{x})|^2\Big)\, d\mathbf{x}\, dt + \frac{1}{4}\int_0^T \int_{\mathbb{R}^2\setminus\mathcal{K}} \nabla\Big(\frac{f(r)}{r}\Big)\cdot\nabla(u(t,\mathbf{x}))^2\, d\mathbf{x}\, dt.
\end{aligned}$$

Here we have again integrated by parts and used the Divergence Theorem along with the Dirichlet boundary conditions. An additional application of the Divergence Theorem then shows that

$$\frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}} \frac{f(r)}{r}u(T,\mathbf{x})\partial_t u(T,\mathbf{x})\,d\mathbf{x} - \frac{1}{2}\int_0^T\int_{\mathbb{R}^2\backslash\mathcal{K}} \frac{f(r)}{r}\Big((\partial_t u(t,\mathbf{x}))^2 - |\nabla u(t,\mathbf{x})|^2\Big)\,d\mathbf{x}\,dt \\ - \frac{1}{4}\int_0^T\int_{\mathbb{R}^2\backslash\mathcal{K}} \nabla\cdot\nabla\Big(\frac{f(r)}{r}\Big)(u(t,\mathbf{x}))^2\,d\mathbf{x}\,dt = \frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}}\frac{f(r)}{r}u(0,\mathbf{x})\partial_t u(0,\mathbf{x})\,d\mathbf{x}. \tag{2.5}$$

Provided that $f(r) \geq 0$, by (1.2), the last term in the left side of (2.4) is nonnegative. By adding (2.3), (2.4), and (2.5) and dropping the nonnegative boundary term in (2.4), we see that

$$\frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}} f(r)\Big[\Big(\partial_t+\partial_r\Big)u(T,\mathbf{x})\Big]^2\,d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}} f(r)|\nabla\!\!\!/\, u(T,\mathbf{x})|^2\,d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}}\frac{f(r)}{r}u(T,\mathbf{x})\partial_t u(T,\mathbf{x})\,d\mathbf{x} \\ + \frac{1}{2}\int_0^T\int_{\mathbb{R}^2\backslash\mathcal{K}} f'(r)\Big[\Big(\partial_t+\partial_r\Big)u(t,\mathbf{x})\Big]^2\,d\mathbf{x}\,dt + \int_0^T\int_{\mathbb{R}^2\backslash\mathcal{K}}\Big(\frac{f(r)}{r}-\frac{1}{2}f'(r)\Big)|\nabla\!\!\!/\, u(t,\mathbf{x})|^2\,d\mathbf{x}\,dt \\ - \frac{1}{4}\int_0^T\int_{\mathbb{R}^2\backslash\mathcal{K}}\Delta\Big(\frac{f(r)}{r}\Big)(u(t,\mathbf{x}))^2\,d\mathbf{x}\,dt \\ \leq \frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}} f(r)\Big[\Big(\partial_t+\partial_r\Big)u(0,\mathbf{x})\Big]^2\,d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}} f(r)|\nabla\!\!\!/\, u(0,\mathbf{x})|^2\,d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}}\frac{f(r)}{r}u(0,\mathbf{x})\partial_t u(0,\mathbf{x})\,d\mathbf{x}. \tag{2.6}$$

In order to get a meaningful estimate, we will need to show that the energy-type contribution on the time slice $t = T$ is nonnegative. To this end, we consider

$$\frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}} f(r)\Big[\Big(\partial_t+\partial_r\Big)u(t,\mathbf{x})\Big]^2\,d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}}\frac{f(r)}{r}u(t,\mathbf{x})\partial_t u(t,\mathbf{x})\,d\mathbf{x} = \frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}} f(r)\Big[\Big(\partial_t+\partial_r\Big)u(t,\mathbf{x})\Big]^2\,d\mathbf{x} \\ + \frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}}\frac{f(r)}{r}u(t,\mathbf{x})\Big(\partial_t+\partial_r\Big)u(t,\mathbf{x})\,d\mathbf{x} - \frac{1}{4}\int_{\mathbb{R}^2\backslash\mathcal{K}}\frac{f(r)}{r}\partial_r(u(t,\mathbf{x}))^2\,d\mathbf{x}.$$

The Divergence Theorem and the boundary conditions give that

$$-\frac{1}{4}\int_{\mathbb{R}^2\backslash\mathcal{K}}\frac{f(r)}{r}\partial_r(u(t,\mathbf{x}))^2\,d\mathbf{x} = \frac{1}{4}\int_{\mathbb{R}^2\backslash\mathcal{K}}\frac{f'(r)}{r}(u(t,\mathbf{x}))^2\,d\mathbf{x}.$$

Hence, if we complete the square, we see that

$$\frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}} f(r)\Big[\Big(\partial_t+\partial_r\Big)u(t,\mathbf{x})\Big]^2\,d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}}\frac{f(r)}{r}u(t,\mathbf{x})\partial_t u(t,\mathbf{x})\,d\mathbf{x} \\ = \frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}} f(r)\Big[\Big(\partial_t+\partial_r+\frac{1}{2r}\Big)u(t,\mathbf{x})\Big]^2\,d\mathbf{x} + \frac{1}{4}\int_{\mathbb{R}^2\backslash\mathcal{K}}\Big(\frac{f'(r)}{r}-\frac{f(r)}{2r^2}\Big)(u(t,\mathbf{x}))^2\,d\mathbf{x}. \tag{2.7}$$

Making this substitution in (2.6) gives

$$\frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}} f(r)\Big[\Big(\partial_t+\partial_r+\frac{1}{2r}\Big)u(T,\mathbf{x})\Big]^2\,d\mathbf{x} + \frac{1}{4}\int_{\mathbb{R}^2\backslash\mathcal{K}}\Big(\frac{f'(r)}{r}-\frac{f(r)}{2r^2}\Big)(u(T,\mathbf{x}))^2\,d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}} f(r)|\nabla\!\!\!/\, u(T,\mathbf{x})|^2\,d\mathbf{x} \\ + \frac{1}{2}\int_0^T\int_{\mathbb{R}^2\backslash\mathcal{K}} f'(r)\Big[\Big(\partial_t+\partial_r\Big)u(t,\mathbf{x})\Big]^2\,d\mathbf{x}\,dt + \int_0^T\int_{\mathbb{R}^2\backslash\mathcal{K}}\Big(\frac{f(r)}{r}-\frac{1}{2}f'(r)\Big)|\nabla\!\!\!/\, u(t,\mathbf{x})|^2\,d\mathbf{x}\,dt \\ - \frac{1}{4}\int_0^T\int_{\mathbb{R}^2\backslash\mathcal{K}}\Delta\Big(\frac{f(r)}{r}\Big)(u(t,\mathbf{x}))^2\,d\mathbf{x}\,dt \leq \frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}} f(r)\Big[\Big(\partial_t+\partial_r+\frac{1}{2r}\Big)u(0,\mathbf{x})\Big]^2\,d\mathbf{x} \\ + \frac{1}{4}\int_{\mathbb{R}^2\backslash\mathcal{K}}\Big(\frac{f'(r)}{r}-\frac{f(r)}{2r^2}\Big)(u(0,\mathbf{x}))^2\,d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^2\backslash\mathcal{K}} f(r)|\nabla\!\!\!/\, u(0,\mathbf{x})|^2\,d\mathbf{x}. \tag{2.8}$$

We seek a function $f \in C^2(\mathbb{R}_+)$ so that the coefficient of each term in the left side is nonnegative. To this end, for $0 \le p \le 1$, we set

$$f(r) = r(\ln r)^p,$$

which is nonnegative on $\mathbb{R}^2\backslash\mathcal{K}$ as $|\mathbf{x}| \ge 1$ for all $\mathbf{x} \in \mathbb{R}^2\backslash\mathcal{K}$. Moreover,

$$f'(r) = (\ln r)^p + p(\ln r)^{p-1} \ge (\ln r)^p,$$

and

$$\frac{2p+1}{2}(\ln r)^p \ge f'(r) - \frac{1}{2}\frac{f(r)}{r} = \frac{1}{2}(\ln r)^p + p(\ln r)^{p-1} \ge \frac{1}{2}(\ln r)^p.$$

Since $|\mathbf{x}| \ge e^2$ on $\mathbb{R}^2\backslash\mathcal{K}$, we additionally have, for example,

$$\begin{aligned} \frac{f(r)}{r} - \frac{1}{2}f'(r) &= \frac{1}{2}(\ln r)^p - \frac{p}{2}(\ln r)^{p-1} \\ &= \frac{1}{2}(\ln r)^{p-1}\Big(\frac{1}{2}\ln r + \Big[\frac{1}{2}\ln r - p\Big]\Big) \\ &\ge \frac{1}{4}(\ln r)^p \quad \text{provided that } p \le 1. \end{aligned}$$

And finally, we note that

$$\begin{aligned} -\frac{1}{4}\Delta\Big(\frac{f(r)}{r}\Big) &= -\frac{1}{4}r^{-1}\partial_r\Big(r\partial_r\Big(\frac{f(r)}{r}\Big)\Big) \\ &= \frac{p(1-p)}{4}r^{-2}(\ln r)^{p-2}, \end{aligned}$$

which is nonnegative if $0 \le p \le 1$. If we make these substitutions in (2.8), the main result (1.4) follows immediately, which completes the proof.

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