ALMOST GLOBAL EXISTENCE FOR SYSTEMS OF NONLINEAR WAVE EQUATIONS IN THREE SPATIAL DIMENSIONS

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ABSTRACT

Taylor Rhoads: Almost global existence for systems of nonlinear wave equations in three spatial dimensions
(Under the direction of Jason Metcalfe)

This work establishes an almost global existence result for systems of semilinear wave equations in three spatial dimensions of the form $\Box u = Q(u, u')$ with sufficiently small initial data. We make the additional assumptions that Q vanishes to second order at the origin and $(\partial_{u^J}\partial_{u^K}Q)(0,0) = 0$ to avoid terms of the form u^2 in the nonlinearity. The result is an extension of the almost global existence result established by Lindblad for scalar equations in three dimensions. To prove this existence statement, we rely on two key components. The first is a new local energy estimate that results from combining an r^p weighted multiplier, inspired by the work of Dafermos and Rodnianski, with a ghost weight $e^{-\sigma(t-r)}$, which was initially used by Alinhac. The second is a Klainerman-Sobelev type estimate from the work of Metcalfe, Tataru, and Tohaneanu, which is used to obtain sufficient decay of solutions to close an iteration argument.

To my parents, Dawn and Roger Rhoads.

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CHAPTER 1

Introduction

This work develops a new method that can be used to prove long-time existence results for nonlinear wave equations with sufficiently small initial data. Here we allow the nonlinearity to depend on the solution itself rather than just its derivatives. There are two key components to this method. The first is a new integrated local energy estimate inspired by the r^p-weighted estimate of Dafermos and Rodnianski [2] and Alinhac's ghost weight method [1]. This estimate will be established in Chapter 2. The second component is a Klainerman-Sobolev type estimate from Metcalfe, Tataru, and Tohaneau [16], which will be introduced in Chapter 3. This estimate will be used to obtain the necessary decay to establish long-time existence. Finally, in Chapter 4, we will apply these tools to establish an almost global existence result for a class of semilinear wave equations in three spatial dimensions.

1.1 Setup

Throughout, we will primarily be concerned with systems of equations of the form

$$\begin{cases} \Box u^I = Q^I(u, u') \\ u^I(0, \cdot) = f^I \\ \partial_t u^I(0, \cdot) = g^I \end{cases} \tag{1.1}$$

where $\Box = \partial_t^2 - \Delta$ is the Euclidean wave operator, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$, and $I = 1, \dots, M$. We denote $u = (u^1, \dots, u^M)$, $f = (f^1, \dots, f^M)$, and $g = (g^1, \dots g^M)$. We reserve the notation ∇u for the spatial gradient of u and use $u' = \partial u = (\partial_t u, \nabla u)$ to represent the full space-time gradient. The components of f and g are assumed to be smooth and compactly supported on $\mathbb{R}^3$. For simplicity, we will take

$$\text{supp } f^I, \text{ supp } g^I \subset \{|x| \le 1\}, \text{ for } I = 1, 2, \dots, M. \tag{1.2}$$

We further assume that each Q^I is a real-valued, smooth function on $\mathbb{R}^2$ that vanishes to second order at the origin. We will require that

$$(\partial_{u^J}\partial_{u^K}Q^I)(0,0) = 0, \ \forall I, J, K = 1, \dots M. \tag{1.3}$$

This prevents terms of the form u^2 from appearing in the quadratic part of Q.

Since higher degree terms will be better behaved with small initial data, in the proof of the main theorem, we will consider

$$Q^I(u,u') = a^{I,\alpha}_{JK} u^J \partial_\alpha u^K + b^{I,\alpha\beta}_{JK} \partial_\alpha u^J \partial_\beta u^K \tag{1.4}$$

for simplicity. Here we use the Einstein summation convention where repeated indices are assumed to be summed. Greek indices are summed from 0 to 3, lowercase Latin indices are summed from 1 to 3 where we take $\partial_0 = \partial_t$ and $\partial_j = \partial_{x_j}$, and uppercase Latin indices are summed from 1 to M.

1.2 Previous Results

It was shown by John and Klainerman in [8] that for quasilinear equations of the form $\Box u = Q(u', u'')$ with initial data $u(0,\cdot) = \varepsilon f$ and $\partial_t u(0,\cdot) = \varepsilon g$ where f and g are compactly supported, a solution $u \in C^\infty([0,T_\varepsilon] \times \mathbb{R}^3)$ exists for $0 < T_\varepsilon = \exp(c/\varepsilon)$ provided $\varepsilon > 0$ is sufficiently small. Here, Q is once again smooth, real-valued, and vanishes to second order at $(0,0)$. Since the lower bound on the lifespan of solutions, T_ε, grows exponentially as the size of the initial data shrinks, we say almost global existence holds for such equations.

When we allow the nonlinearity to also depend on the solution u instead of merely derivatives of u, Hörmander [7] proved almost global existence of solutions for scalar equations of the form $\Box u = Q(u, u', u'')$ in four spatial dimensions. With the additional assumption $(\partial^2_{uu}Q)(0,0,0) = 0$, [7] also shows global existence for equations with sufficiently small initial data. With this same assumption in three spatial dimensions, Lindblad [11] proved almost global existence of solutions. Without the assumption, [11] establishes a lower bound of c/ε^2 on the lifespan of solutions.

To establish such results, energy methods are typically employed. Since better estimates are available for derivatives of the solution, in the case of [11] and [7], their results relied on writing lower order terms in the nonlinearity as

$$u\partial u = \frac{1}{2}\partial u^2.$$

As such, one can obtain bounds involving u rather than just ∂u. However, for systems of equations, the lower order terms will be of the form $u^K \partial u^J$. Due to this coupling of equations, we are unable to write such terms in divergence form as in the scalar case. Metcalfe and Morgan [12] were able to remedy this issue for systems in four spatial dimensions and established global existence under the assumption (1.3). Unfortunately, wave equation solutions experience less dispersion in lower dimensions, so the methods in [12] cannot be easily extended to three spatial dimensions. The new method presented in this work will be used to fill this gap in the literature.

1.3 Main Theorem

The main result that we will establish is as follows:

Theorem 1.1. *Assume Q is a smooth function that vanishes to second order and satisfies (1.3). Further assume $f, g \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^M)$. Then for constants $\varepsilon_0 > 0$ and $c > 0$ sufficiently small and $N \in \mathbb{N}$ sufficiently large, if $\varepsilon \le \varepsilon_0$ and*

$$\sum_{|\alpha| \le N+1} \|\partial^\alpha f\|_{L^2} + \sum_{|\alpha| \le N} \|\partial^\alpha g\|_{L^2} \le \varepsilon, \tag{1.5}$$

there exists a unique solution $u \in (C^\infty([0, T_\varepsilon] \times \mathbb{R}^3))^M$ to (1.1) for

$$T_\varepsilon = \exp\left(\frac{c}{\varepsilon^{1/3}}\right). \tag{1.6}$$

In the proof of Theorem 1.1, we will assume for simplicity that f and g are supported in $\{x \in \mathbb{R}^3 : \ |x| \le 1\}$. With this assumption, the constants ε_0, N, and c are independent of the functions f, g, and Q.

It is worth noting that this result can be extended to quasilinear systems of the form $\Box u^I = Q^I(u, u', u'')$ under the assumption (1.3) as was done in [7], [11], and [12]. To do so, we will require a modification of the new local energy estimate presented in Chapter 2 as well as the perturbation of the standard local energy estimate that was established in [13]. The details of this extension have been omitted from this work.

Unfortunately, due to the logarithmic loss in the local energy estimate (2.5) in Chapter 2, there is a difference between the lower bound on the lifespan for almost global existence in Theorem 1.1 compared to that of [11]. It is not yet clear what the sharp power on the ε in (1.6) should be for systems.

1.4 Notation

Throughout we will take $a \lesssim b$ to mean $a \leq C \cdot b$ for some constant C where the implicit constant may change from line to line. Additionally, $a \approx b$ will mean both $a \lesssim b$ and $b \lesssim a$ hold.

For the spatial gradient ∇, we may use the orthogonal decomposition

$$\nabla = \frac{x}{r}\partial_r + \not\nabla$$

where r is the radial spherical coordinate, $\partial_r = \frac{x_i}{r}\partial_i$ is the radial component of the derivative, and $\not\nabla$ denotes the angular component. As such, any space-time gradient may be bounded using

$$|\partial w| \lesssim |\not\nabla w| + |(\partial_t + \partial_r)w| + |(\partial_t - \partial_r)w| \tag{1.7}$$

for any appropriate function w.

We further note that better decay properties are available for derivatives of the form $\not\nabla$ and $\partial_t + \partial_r$ as these derivatives are tangent to the light cone $\{r = t\}$. These derivatives both appear in the new local energy estimate presented in Chapter 2. As such, we will refer to these derivatives as "good derivatives" and use the notation

$$\not\partial = (\not\nabla, \partial_t + \partial_r)$$

to denote the collection of good derivatives. A "bad derivative" will be of the form $\partial_t - \partial_r$. In the proof of Theorem 1.1, more care will be required with the appearance of bad derivatives.

1.4.1 Invariant Vector Fields

To obtain the appropriate decay for the proof of Theorem 1.1, we will rely on a class of vector fields that are invariant with respect to the wave equation. That is, they will preserve the homogeneous wave equation. These vector fields were originally used by Klainerman [10] to establish an $L^\infty - L^2$

estimate known as the Klainerman-Sobolev inequality, which he used to prove long-time existence for certain classes nonlinear wave equations. They have since become a convenient tool in this field.

The invariant vector fields include the generators of space-time translations

$$\partial_0, \partial_1, \partial_2, \partial_3$$

and the generators of spatial rotations

$$\Omega_{jk} = x_k\partial_j - x_j\partial_k \ \text{ for } 1 \leq j < k \leq 3.$$

The collection of rotational vector fields will further be denoted by $\Omega = (\Omega_{32}, \Omega_{13}, \Omega_{21})$ so that $\Omega = x \times \nabla$. We will also use the scaling vector field

$$S = t\partial_t + r\partial_r.$$

The full collection of vector fields will be denoted

$$\Gamma = (\Gamma_1, \ldots, \Gamma_8),$$

and we will rely on multi-index notation. That is, for an index $\alpha = (\alpha_1, \ldots, \alpha_8)$ with $\alpha_i \in \mathbb{N}$ for each $i = 1, \ldots, 8$, we take

$$\Gamma^\alpha = \Gamma_1^{\alpha_1} \cdots \Gamma_8^{\alpha_8}.$$

Additionally, for a positive integer N, we denote

$$|\Gamma^{\leq N} w| = \sum_{|\alpha| \leq N} |\Gamma^\alpha w|, \ \ |\partial^{\leq N} w| = \sum_{|\alpha| \leq N} |\partial^\alpha w|.$$

The following identity for the scaling vector field will be beneficial in establishing bounds for terms involving derivatives.

$$S = \frac{1}{2}\left[(t-r)(\partial_t - \partial_r) + (t+r)(\partial_t + \partial_r)\right]. \tag{1.8}$$

Furthermore, from the identity

$$\not\nabla = -\frac{x}{r^2} \times \Omega,$$

we may bound

$$|\not\nabla w| \le r^{-1}|\Omega w|. \tag{1.9}$$

It is worth noting that Klainerman's work required additional vector fields corresponding to Lorentz boosts while the methods presented here do not. This will allow for easier application to settings such as exterior domain problems or asymptotically flat geometries where boosts do not behave as nicely. In fact, one could easily extend Theorem 1.1 to the corresponding Dirichlet wave equation exterior to a star-shaped obstacle, for example, as was done in [12]. The results of [8], [7] and [11] have largely already been extended to exterior domain problems in [9], [3], [4], and [14]. See also [6] and [5].

1.4.2 Useful Commutators

For convenience, we record various commutation properties of the invariant vector fields. First,

$$[\Box, \partial] = 0, \quad [\Box, \Omega] = 0, \quad [\Box, S] = 2\Box.$$

As such,

$$|\Box\Gamma^\alpha u| \lesssim |\Gamma^{\le|\alpha|}\Box u|. \tag{1.10}$$

We also note that

$$[\nabla, \partial_r] = [\not\nabla, \partial_r] = \frac{1}{r}\not\nabla, \tag{1.11}$$

which will be used in the proof of the local energy estimates in Chapter 2.

Next, we record the commutators of the vector fields amongst themselves. We have

$$[\partial, S] = \partial, \quad [S, \Omega] = 0, \quad [\partial_t, \Omega] = 0. \tag{1.12}$$

Additionally, for each $i = 1, 2, 3$ and $1 \le j < k \le 3$,

$$[\partial_i, \Omega_{jk}] = \delta_{ik}\partial_j - \delta_{ij}\partial_k. \tag{1.13}$$

Therefore, for each $1 \leq i, j \leq 8$, there exist constants a_{ij}^k so that

$$[\Gamma_i, \Gamma_j] = a_{ij}^k \Gamma_k.$$

Furthermore,

$$|[\Gamma, \partial]w| \lesssim |\partial w|. \tag{1.14}$$

Lastly, it will be helpful to record commutators of the vector fields with both good and bad derivatives. For the scaling vector field, we find

$$[\not\nabla, S] = \not\nabla, \;\; [\partial_t + \partial_r, S] = \partial_t + \partial_r, \;\; [\partial_t - \partial_r, S] = \partial_t - \partial_r. \tag{1.15}$$

For the rotational vector field, for each $i = 1, 2, 3$ and $1 \leq j < k \leq 3$, one computes

$$[\not\nabla_i, \Omega_{jk}] = \delta_{ik} \not\nabla_j - \delta_{ij} \not\nabla_k \tag{1.16}$$

and

$$[\partial_t + \partial_r, \Omega] = 0, \;\; [\partial_t - \partial_r, \Omega] = 0. \tag{1.17}$$

Finally, we record

$$[\not\nabla, \partial_t] = 0, \;\; [\partial_t + \partial_r, \partial_t] = 0, \;\; [\partial_t - \partial_r, \partial_t] = 0, \tag{1.18}$$

as well as

$$|[\not\nabla, \nabla]w|| \lesssim \frac{1}{r} |\not\nabla w|,$$

and for $j = 1, 2, 3$,

$$[\partial_t + \partial_r, \partial_j] = -\frac{1}{r} \not\nabla_j, \;\; [\partial_t - \partial_r, \partial_j] = \frac{1}{r} \not\nabla_j. \tag{1.19}$$

1.4.3 Dyadic Regions

Since our initial data are assumed to be supported where $|x| \leq 1$, by finite speed of propagation, it suffices to consider the region $C = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3 : r \leq t + 1\}$ where $r = |x|$. We begin with a dyadic decomposition of C in time. For each $T \geq 0$, we define

$$C_T = \{(t,x) : T \leq t \leq 2T\} \cap C.$$

From here, we will need a further dyadic decomposition in r away from the light cone, $\{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 : t = |x|\}$, and a decomposition in $|t-r|$ close to the light cone.

To this end, for $R > 1$, we define

$$C_T^R = \{(t,x) : R < r < 2R\} \cap C_T$$

and for $R = 1$,

$$C_T^{R=1} = \{(t,x) : r < 2\} \cap C_T.$$

Similarly, for $U > 1$, we take

$$C_T^U = \{(t,x) : U < t - r < 2U\} \cap C_T$$

and

$$C_T^{U=1} = \{(t,x) : |t-r| < 2\} \cap C_T.$$

We will use a tilde to denote enlargements of these sets on both of their respective scales. For example, for $R > 1$, we may use

$$\tilde{C}_T^R = \left\{(t,x) : \frac{7T}{8} < t < \frac{17T}{8}, \frac{7R}{8} < r < \frac{17R}{8}\right\} \cap C.$$

At times further enlargements will be required, which will be denoted $\tilde{\tilde{C}}_T^R$ and $\tilde{\tilde{C}}_T^U$.

It is important to note that

$$C_T = \left(\bigcup_{1 \leq R \leq T/4} C_T^R\right) \cup \left(\bigcup_{1 \leq U \leq T/4} C_T^U\right)$$

so that, for a suitable function w, we may write

$$\int_0^T \int_{\mathbb{R}^3} |w(t,x)| dx dt \lesssim \sum_{\tau \le T} \left(\sum_{R \le \tau/4} \int\int_{C_\tau^R} |w(t,x)| dx dt + \sum_{U \le \tau/4} \int\int_{C_\tau^U} |w(t,x)| dx dt \right) \tag{1.20}$$

where all sums and unions are assumed to be over dyadic values in τ, R, and U.

In later arguments, we require a dyadic decomposition of $[0,T] \times \mathbb{R}^3$ in $|t-r|$. To this end, for $U > 1$, we define

$$X_U = \{(t,x) \in [0,T] \times \mathbb{R}^3 : U < |t-r| < 2U\} \tag{1.21}$$

and for $U = 1$,

$$X_{U=1} = \{(t,x) \in [0,T] \times \mathbb{R}^3 : |t-r| < 2\}. \tag{1.22}$$

Additionally, we may create a dyadic decomposition of $\mathbb{R}^3$ in r using

$$A_R = \{x \in \mathbb{R}^3 : R < r < 2R\}$$

for $R > 1$ and

$$A_{R=1} = \{x \in \mathbb{R}^3 : r < 2\}.$$

Once again, we will use $\tilde{X}_U$ and $\tilde{A}_R$ to denote the respective enlargements of these regions.

The notation $\langle \cdot \rangle$ will denote a smooth function on $\mathbb{R}$ so that $\langle r \rangle \ge 3$ and $\langle r \rangle \approx r$ for $r \gg 1$. For example, it would suffice to use $\langle r \rangle = 3 + r$. It is important to note on C_T^R, $\tilde{C}_T^R$, and $\tilde{\tilde{C}}_T^R$ with $1 \le R \le T/4$, we have

$$\langle r \rangle \approx R, \;\; t - r \approx T,$$

and on C_T^U, $\tilde{C}_T^U$, and $\tilde{\tilde{C}}_T^U$ with $1 \le U \le T/4$,

$$r \approx T, \;\; \langle t - r \rangle \approx U.$$

1.4.4 Norms

Finally, we define the norms to be used. $L^p L^q$ will simply be used in place of $L^p([0,T]; L^q(\mathbb{R}^3)) = L_t^p L_x^q([0,T] \times \mathbb{R}^3)$. Additionally, we will use the following local energy norms as were introduced in

[19] and [16]. For appropriate function u on $\mathbb{R}_+ \times \mathbb{R}^3$, we define

$$\|u\|_{LE} = \sup_{j\geq 0} 2^{-j/2} \|u\|_{L^2L^2([0,T]\times A_{2^j})}$$

and

$$\|u\|_{LE^1} = \|(\partial u, |x|^{-1}u)\|_{LE}.$$

These norms, paired with the standard integrated local energy estimate (2.1), will be a useful tool to bound energy terms with a large number of vector fields in the proof of Theorem 1.1 in Chapter 4.

CHAPTER 2

Local Energy Estimates

One of the key ingredients that will be used to prove Theorem 1.1 is a new local energy estimate. More standard integrated local energy estimates of the form

$$\|\partial u\|_{L^\infty L^2}^2 + \|u\|_{LE^1}^2 \lesssim \|\partial u(0,\cdot)\|_{L^2}^2 + \int_0^T \int |\Box u| \left(|\partial u| + \frac{|u|}{r} \right) dxdt \tag{2.1}$$

have been used by others such as Keel, Smith, and Sogge [9] to establish long-time existence results for classes of wave equations. While there have been many contributors to these standard local energy estimates, their origins can be traced back to Morawetz [17]. To prove (2.1), one pairs $\Box u$ with a multiplier of the form $\left(\frac{r}{r+R}\right)\partial_r u + \frac{u}{r+R}$ as was done by Rodnianski [18, Appendix]. The result follows by integrating over $[0,T]\times\mathbb{R}^3$ and using integration by parts. For a more complete survey of wave equation results using local energy estimates on asymptotically flat space-times, see [15].

Modifications to this standard local energy estimate can be established by making different choices for the multiplier. Dafermos and Rodnianski [2] have used r^p-weighted multipliers, and Alinhac [1] can be credited with using multipliers that contain factors of the form $e^{-\sigma(t-r)}$. Alinhac's method is often referred to as the "ghost-weight" method because for appropriate choices of the function σ, the exponential factor $e^{-\sigma}$ may be bounded by positive constants from above and below and thus, effectively disappears from the estimate altogether. Both methods give rise to estimates involving the good derivatives only.

Our new local energy estimate, given below in Proposition 2.1, will be established using a method that combines the r^p-weighted method of Dafermos and Rodnianski along with Alinhac's ghost weight method. Specifically, to prove the estimate, we will use a multiplier of the form $e^{-\sigma(t-r)}r^p\left(\partial_t + \partial_r + \frac{1}{r}\right)u$. We note that a novel feature of this estimate is that it is well suited to be applied in regions localized in either r or $t-r$ depending on proximity to $\{t=r\}$.

Proposition 2.1. *Suppose $T > 0$, $u \in C^2([0,T] \times \mathbb{R}^3)$ and $\sigma \in C^1(\mathbb{R})$. Assume u has compactly supported initial data and that σ is bounded and nondecreasing. Then, for a fixed constant $0 < p < 1$,*

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)}\sigma'(t-r)r^{p-2}u^2\,dxdt + \int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)}r^{p-3}u^2\,dxdt \\
&\qquad + \int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)}r^{p-1}|\nabla\!\!\!/ u|^2\,dxdt + \int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)}r^{p-1}[(\partial_t+\partial_r)u]^2\,dxdt \\
&\qquad\qquad + \int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)}\sigma'(t-r)r^{p-2}[(\partial_t+\partial_r)(ru)]^2\,dxdt \\
&\quad + \sup_{t\in[0,T]}\Big(\int_{\mathbb{R}^3} e^{-\sigma(t-r)}r^{p-2}[(\partial_t+\partial_r)(ru(t,x))]^2\,dx + \int_{\mathbb{R}^3} e^{-\sigma(t-r)}r^{p}|\nabla\!\!\!/ u(t,x)|^2\,dx \\
&\quad\quad + \int_{\mathbb{R}^3} e^{-\sigma(t-r)}\sigma'(t-r)r^{p-1}u(t,x)^2\,dx + \int_{\mathbb{R}^3} e^{-\sigma(t-r)}r^{p-2}u(t,x)^2\,dx\Big) \\
&\lesssim \int_{\mathbb{R}^3} e^{-\sigma(-r)}r^{p-2}[(\partial_t+\partial_r)(ru(0,x))]^2\,dx + \int_{\mathbb{R}^3} e^{-\sigma(-r)}r^{p}|\nabla\!\!\!/ u(0,x)|^2\,dx \\
&\quad\quad + \int_{\mathbb{R}^3} e^{-\sigma(-r)}\sigma'(-r)r^{p-1}u(0,x)^2\,dx + \int_{\mathbb{R}^3} e^{-\sigma(-r)}r^{p-2}u(0,x)^2\,dx \\
&\qquad\qquad\qquad + \int_0^T \int_{\mathbb{R}^3} \Box u\, e^{-\sigma(t-r)}r^{p}\left(\partial_t+\partial_r+\frac{1}{r}\right)u\,dxdt. \qquad (2.2)
\end{aligned}$$

The analogous result for $p = 1$ is

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} (\log\langle r\rangle)^{-2}e^{-\sigma(t-r)}\sigma'(t-r)r^{-1}u^2\,dxdt + \int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)}r^{-2}u^2\,dxdt \\
&\qquad + \int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)}|\nabla\!\!\!/ u|^2\,dxdt + \int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)}[(\partial_t+\partial_r)u]^2\,dxdt \\
&\qquad\qquad + \int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)}\sigma'(t-r)r^{-1}[(\partial_t+\partial_r)(ru)]^2\,dxdt \\
&\quad + \sup_{t\in[0,T]}\Big(\int_{\mathbb{R}^3} e^{-\sigma(t-r)}r^{-1}[(\partial_t+\partial_r)(ru(t,x))]^2\,dx + \int_{\mathbb{R}^3} e^{-\sigma(t-r)}r|\nabla\!\!\!/ u(t,x)|^2\,dx \\
&\quad\quad + \int_{\mathbb{R}^3} (\log\langle r\rangle)^{-1}e^{-\sigma(t-r)}\sigma'(t-r)u(t,x)^2\,dx + \int_{\mathbb{R}^3} e^{-\sigma(t-r)}r^{-1}u(t,x)^2\,dx\Big) \\
&\lesssim \int_{\mathbb{R}^3} e^{-\sigma(-r)}r^{-1}[(\partial_t+\partial_r)(ru(0,x))]^2\,dx + \int_{\mathbb{R}^3} e^{-\sigma(-r)}r|\nabla\!\!\!/ u(0,x)|^2\,dx \\
&\quad\quad + \int_{\mathbb{R}^3} (\log\langle r\rangle)^{-1}e^{-\sigma(-r)}\sigma'(-r)u(0,x)^2\,dx + \int_{\mathbb{R}^3} e^{-\sigma(-r)}r^{-1}u(0,x)^2\,dx \\
&\qquad\qquad\qquad + \|u\|_{LE^1}^2 + \int_0^T \int_{\mathbb{R}^3} \Box u\, e^{-\sigma(t-r)}r\left(\partial_t+\partial_r+\frac{1}{r}\right)u\,dxdt. \qquad (2.3)
\end{aligned}$$

For completeness, this result is recorded for a range of p values. However, the proof of Theorem 1.1 will only require the case where $p = 1$. In the following corollary, we record the result with $p = 1$

after bounding exponential factors by constants, writing terms using norms where applicable, and making an appropriate choice for the function σ. The identity

$$\left(\partial_t + \partial_r + \frac{1}{r}\right) u = \frac{1}{r}(\partial_t + \partial_r)(ru) \tag{2.4}$$

will also be useful.

In the remainder of this chapter, we will first state and prove the corollary in the case where $p = 1$. Subsequently, we will establish several lemmas. Finally, these lemmas will be used in order to prove Proposition 2.1.

Corollary 2.2. *Suppose $T > 0$ and $u \in C^2([0,T] \times \mathbb{R}^3)$ with compactly supported initial data. For each $U > 0$, define the function*

$$\sigma_U(s) = \frac{s}{|s| + U}.$$

Then,

$$\begin{aligned}
&\|\langle r\rangle^{1/2}\not{\partial} u\|^2_{L^\infty L^2} + \|r^{-1/2}u\|^2_{L^\infty L^2} + \|\not{\partial} u\|^2_{L^2L^2} + \|r^{-1}u\|^2_{L^2L^2} \\
&\quad + \sup_{U\geq 1}\left(\|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(ru)\|^2_{L^2L^2(X_U)} + \|r^{-1/2}(\log\langle r\rangle)^{-1}\langle t-r\rangle^{-1/2}u\|^2_{L^2L^2(X_U)}\right) \\
&\qquad \lesssim \|\not{\partial} u(0,\cdot)\|^2_{L^2} + \|r^{-1/2}u(0,\cdot)\|^2_{L^2} + \|u\|^2_{LE^1} + \|\partial u\|^2_{L^\infty L^2} \\
&\qquad\qquad + \sup_{U\geq 1}\sup_{t\in[0,T]}\left|\int_0^t\int e^{-\sigma_U(t-r)}\Box u\cdot(\partial_t+\partial_r)(ru)\,dxdt\right|.
\end{aligned} \tag{2.5}$$

Proof. We begin by applying Proposition 2.1 with $p = 1$. Note that σ_U bounded, so exponential factors can be bounded by a constant. Therefore,

$$\begin{aligned}
&\|(\log\langle r\rangle)^{-1}\sigma_U'(t-r)^{1/2}r^{-1/2}u\|_{L^2L^2}^2+\|r^{-1}u\|_{L^2L^2}^2+\|\not\nabla u\|_{L^2L^2}^2\\
&\qquad+\|\sigma_U'(t-r)^{1/2}r^{-1/2}(\partial_t+\partial_r)(ru)\|_{L^2L^2}^2+\|(\partial_t+\partial_r)u\|_{L^2L^2}^2\\
&\qquad+\sup_{t\in[0,T]}\Big(\|(\log\langle r\rangle)^{-1/2}\sigma_U'(t-r)^{1/2}u(t,\cdot)\|_{L^2}^2+\|r^{-1/2}u(t,\cdot)\|_{L^2}^2\\
&\qquad+\|r^{1/2}\not\nabla u(t,\cdot)\|_{L^2}^2+\|r^{-1/2}(\partial_t+\partial_r)(ru(t,\cdot))\|_{L^2}^2\Big)\\
&\lesssim\|(\log\langle r\rangle)^{-1/2}\sigma_U'(-r)^{1/2}u(0,\cdot)\|_{L^2}^2+\|r^{-1/2}u(0,\cdot)\|_{L^2}^2+\|r^{1/2}\not\nabla u(0,\cdot)\|_{L^2}^2\\
&\qquad+\|r^{-1/2}(\partial_t+\partial_r)(ru(0,\cdot))\|_{L^2}^2+\sup_{t\in[0,T]}\left|\int_0^t\int_{\mathbb{R}^3}\Box u e^{-\sigma_U(t-r)}r\left(\partial_t+\partial_r+\frac{1}{r}\right)u\,dxdt\right| \qquad (2.6)
\end{aligned}$$

where all L^2L^2 and L^2 norms are over $[0,T]\times\mathbb{R}^3$ and $\mathbb{R}^3$, respectively.

For the function σ_U,

$$\sigma_U'(s)=\frac{U}{(|s|+U)^2}.$$

Therefore, on the regions X_U as in (1.21),

$$\sigma_U'(t-r)\gtrsim\langle t-r\rangle^{-1}.$$

It follows that

$$\begin{aligned}
&\|r^{-1/2}(\log\langle r\rangle)^{-1}\langle t-r\rangle^{-1/2}u\|_{L^2L^2(X_U)}^2+\|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(ru)\|_{L^2L^2(X_U)}^2\\
&\qquad\lesssim\|(\log\langle r\rangle)^{-1}\sigma_U'(t-r)^{1/2}r^{-1/2}u\|_{L^2L^2}^2+\|\sigma_U'(t-r)^{1/2}r^{-1/2}(\partial_t+\partial_r)(ru)\|_{L^2L^2}^2. \qquad (2.7)
\end{aligned}$$

For the L^2 norms at a fixed time t on the left hand side of (2.6), we estimate

$$\begin{aligned}
&\|\langle r\rangle^{1/2}\not\partial u\|_{L^\infty L^2}^2+\|r^{-1/2}u\|_{L^\infty L^2}^2\lesssim\sup_{t\in[0,T]}\Big(\|(\log\langle r\rangle)^{-1/2}\sigma_U'(t-r)^{1/2}u(t,\cdot)\|_{L^2}^2\\
&\qquad+\|r^{-1/2}u(t,\cdot)\|_{L^2}^2+\|r^{1/2}\not\nabla u(t,\cdot)\|_{L^2}^2+\|r^{-1/2}(\partial_t+\partial_r)(ru(t,\cdot))\|_{L^2}^2\Big). \qquad (2.8)
\end{aligned}$$

Also, since the initial data for u is compactly supported, we have

$$\|(\log\langle r\rangle)^{-1/2}\sigma_U'(-r)^{1/2}u(0,\cdot)\|_{L^2}^2+\|r^{-1/2}u(0,\cdot)\|_{L_x^2}^2+\|r^{1/2}\not\nabla u(0,\cdot)\|_{L^2}^2+\|r^{-1/2}(\partial_t+\partial_r)(ru(0,\cdot))\|_{L^2}^2$$
$$\lesssim \|\not\partial u(0,\cdot)\|_{L^2}^2+\|r^{-1/2}u(0,\cdot)\|_{L^2}^2. \quad (2.9)$$

Finally, (2.5) is established using (2.6), (2.7), (2.8), (2.9), and (2.4), and by taking a supremum over U. □

Before we present the proof Proposition 2.1, it will be beneficial to establish several lemmas. First, Lemma 2.3 serves as the foundation for our local energy estimate as it relates the term

$$\int_0^T\int_{\mathbb{R}^3}\Box u e^{-\sigma(t-r)}r^p\left(\partial_t+\partial_r+\frac{1}{r}\right)u\,dxdt$$

to terms involving weighted energy of u. We will prove this lemma using a series of integration by parts arguments.

Lemma 2.3. *Suppose $T>0$, $u\in C^2([0,T]\times\mathbb{R}^3)$ and $\sigma\in C^1(\mathbb{R})$. Assume u has compactly supported initial data. Then for each fixed constant $p>0$,*

$$\int_0^T\int_{\mathbb{R}^3}e^{-\sigma(t-r)}(2\sigma'(t-r)r^{p-2}+pr^{p-3})[(\partial_t+\partial_r)(ru)]^2dxdt+(2-p)\int_0^T\int_{\mathbb{R}^3}e^{-\sigma(t-r)}r^{p-1}|\not\nabla u|^2dxdt$$
$$=-\int_{\mathbb{R}^3}e^{-\sigma(t-r)}r^{p-2}[(\partial_t+\partial_r)(ru)]^2\,dx\Big|_0^T-\int_{\mathbb{R}^3}e^{-\sigma(t-r)}r^p|\not\nabla u|^2\,dx\Big|_0^T$$
$$+2\int_0^T\int_{\mathbb{R}^3}\Box u e^{-\sigma(t-r)}r^p\left(\partial_t+\partial_r+\frac{1}{r}\right)u\,dxdt. \quad (2.10)$$

Proof. Suppose u and σ are as above. Using the identity

$$\Box u=r^{-1}(\partial_t^2-\partial_r^2-\not\nabla\cdot\not\nabla)(ru) \quad (2.11)$$

along with (2.4), we are able to write

$$\int_0^T\int_{\mathbb{R}^3}\Box u e^{-\sigma(t-r)}r^p\left(\partial_t+\partial_r+\frac{1}{r}\right)u\,dxdt$$
$$=\int_0^T\int_{\mathbb{R}^3}(\partial_t^2-\partial_r^2-\not\nabla\cdot\not\nabla)(ru)e^{-\sigma(t-r)}r^{p-2}(\partial_t+\partial_r)(ru)\,dxdt. \quad (2.12)$$

We will first consider

$$\int_0^T \int_{\mathbb{R}^3} (\partial_t^2 - \partial_r^2)(ru) e^{-\sigma(t-r)} r^{p-2} (\partial_t + \partial_r)(ru)\, dxdt,$$

which can be rewritten as

$$\frac{1}{2} \int_0^T \int_{\mathbb{S}^2} \int_0^\infty (\partial_t - \partial_r) \left([(\partial_t + \partial_r)(ru)]^2 \right) e^{-\sigma(t-r)} r^p \, drd\omega dt$$

after factoring $\partial_t^2 - \partial_r^2$ as $(\partial_t - \partial_r)(\partial_t + \partial_r)$ and converting to spherical coordinates. Next, we will use an integration by parts argument in $\partial_t - \partial_r$ to obtain

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^3} (\partial_t^2 - \partial_r^2)(ru) e^{-\sigma(t-r)} r^{p-2} (\partial_t + \partial_r)(ru)\, dxdt \\
= \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} [(\partial_t + \partial_r)(ru)]^2 e^{-\sigma(t-r)} (2r^{p-2}\sigma'(t-r) + pr^{p-3})\, dxdt \\
+ \frac{1}{2} \int_{\mathbb{R}^3} [(\partial_t + \partial_r)(ru)]^2 e^{-\sigma(t-r)} r^{p-2}\, dx \Big|_0^T. \quad (2.13)
\end{aligned}$$

An additional integration by parts argument yields

$$\int_0^T \int_{\mathbb{R}^3} \not\nabla \cdot \not\nabla (ru) e^{-\sigma(t-r)} r^{p-2} (\partial_t + \partial_r)(ru)\, dxdt = -\int_0^T \int_{\mathbb{R}^3} \not\nabla (ru) \cdot e^{-\sigma(t-r)} r^{p-2} \nabla (\partial_t + \partial_r)(ru)\, dxdt.$$

Using (1.11), one may compute

$$\not\nabla (ru) \cdot \nabla (\partial_t + \partial_r)(ru) = \frac{1}{2} (\partial_t + \partial_r)(|\not\nabla (ru)|^2) + r|\not\nabla u|^2$$

so that

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^3} \not\nabla \cdot \not\nabla (ru) e^{-\sigma(t-r)} r^{p-2} (\partial_t + \partial_r)(ru)\, dxdt \\
= -\frac{1}{2} \int_0^T \int_{\mathbb{R}^3} (\partial_t + \partial_r) \left(|\not\nabla (ru)|^2 \right) e^{-\sigma(t-r)} r^{p-2}\, dxdt - \int_0^T \int_{\mathbb{R}^3} |\not\nabla u|^2 e^{-\sigma(t-r)} r^{p-1}\, dxdt. \quad (2.14)
\end{aligned}$$

A further application of integration by parts on the operator $\partial_t + \partial_r$ gives

$$\int_0^T \int_{\mathbb{R}^3} (\partial_t + \partial_r)\left(|\nabla\!\!\!/(ru)|^2\right) e^{-\sigma(t-r)} r^{p-2}\, dxdt$$
$$= -\int_0^T \int_{\mathbb{R}^3} |\nabla\!\!\!/ u|^2 e^{-\sigma(t-r)} p r^{p-1}\, dxdt + \int_{\mathbb{R}^3} |\nabla\!\!\!/ u|^2 e^{-\sigma(t-r)} r^p\, dx\Big|_0^T. \quad (2.15)$$

We combine equations (2.14) and (2.15) to obtain

$$\int_0^T \int_{\mathbb{R}^3} \nabla\!\!\!/ \cdot \nabla\!\!\!/(ru) e^{-\sigma(t-r)} r^{p-2} (\partial_t + \partial_r)(ru)\, dxdt$$
$$= \left(\frac{p}{2} - 1\right) \int_0^T \int_{\mathbb{R}^3} |\nabla\!\!\!/ u|^2 e^{-\sigma(t-r)} r^{p-1}\, dxdt - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla\!\!\!/ u|^2 e^{-\sigma(t-r)} r^p\, dx\Big|_0^T. \quad (2.16)$$

Thus, equations (2.12), (2.13), and (2.16) prove the lemma. □

In order to incorporate weighted lower-order terms into our local energy estimate, we use Lemmas 2.4 and 2.5. These lemmas provide space-time variants of more standard Hardy inequalities. To avoid singularities from a logarithm, we will make use of a cutoff function $\beta \in C^\infty(\mathbb{R})$ such that $\beta \equiv 0$ for $r < 2$ and $\beta \equiv 1$ for $r > 4$.

Lemma 2.4. *Suppose $T > 0$, $u \in C^2([0,T] \times \mathbb{R}^3)$ and $\sigma \in C^1(\mathbb{R})$. Assume u has compactly supported initial data and σ is nondecreasing. Then, for a fixed constant $0 < p < 1$,*

$$\int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)} \sigma'(t-r) r^{p-2} u^2\, dxdt + \sup_{t\in[0,T]} \int_{\mathbb{R}^3} e^{-\sigma(t-r)} \sigma'(t-r) r^{p-1} u(t,x)^2\, dx$$
$$\lesssim \int_{\mathbb{R}^3} e^{-\sigma(-r)} \sigma'(-r) r^{p-1} u(0,x)^2\, dx + \int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)} \sigma'(t-r) r^{p-2} [(\partial_t + \partial_r)(ru)]^2\, dxdt. \quad (2.17)$$

The analogous result for $p = 1$ is

$$\int_0^T \int_{\mathbb{R}^3} (\log\langle r\rangle)^{-2} e^{-\sigma(t-r)} \sigma'(t-r) r^{-1} u^2\, dxdt + \sup_{t\in[0,T]} \int_{\mathbb{R}^3} \beta(r) (\log\langle r\rangle)^{-1} e^{-\sigma(t-r)} \sigma'(t-r) u(t,x)^2\, dx$$
$$\lesssim \int_{\mathbb{R}^3} (\log\langle r\rangle)^{-1} e^{-\sigma(-r)} \sigma'(-r) u(0,x)^2\, dx + \int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)} \sigma'(t-r) r^{-1} [(\partial_t + \partial_r)(ru)]^2\, dxdt$$
$$+ \|u\|_{LE^1}^2. \quad (2.18)$$

Proof. We begin with the proof of equation (2.17). For a constant $p<1$, since

$$\frac{1}{p-1}(\partial_t+\partial_r)\left(e^{-\sigma(t-r)}\sigma'(t-r)r^{p-1}\right)=e^{-\sigma(t-r)}\sigma'(t-r)r^{p-2},$$

we use integration by parts in $(\partial_t+\partial_r)$ to obtain

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^3} e^{-\sigma(t-r)}\sigma'(t-r)r^{p-2}u^2\,dxdt\\
&\qquad=\frac{2}{1-p}\int_0^T\int_{\mathbb{R}^3} e^{-\sigma(t-r)}\sigma'(t-r)r^{p-2}(\partial_t+\partial_r)(ru)\cdot u\,dxdt\\
&\qquad\qquad+\frac{1}{p-1}\int_{\mathbb{R}^3} e^{-\sigma(t-r)}\sigma'(t-r)r^{p-1}u^2\,dxdt\Big|_0^T\\
&\qquad\leq\frac{2}{1-p}\left(\int_0^T\int_{\mathbb{R}^3} e^{-\sigma(t-r)}\sigma'(t-r)r^{p-2}[(\partial_t+\partial_r)(ru)]^2\,dxdt\right)^{1/2}\\
&\qquad\qquad\cdot\left(\int_0^T\int_{\mathbb{R}^3} e^{-\sigma(t-r)}\sigma'(t-r)r^{p-2}u^2\,dxdt\right)^{1/2}\\
&\qquad\qquad\qquad+\frac{1}{p-1}\int_{\mathbb{R}^3} e^{-\sigma(t-r)}\sigma'(t-r)r^{p-1}u^2\,dxdt\Big|_0^T
\end{aligned}\tag{2.19}$$

where the inequality follows from applying the Schwarz inequality. After moving the time boundary term at time $t=T$ to the left, we may bootstrap the term

$$\int_0^T\int_{\mathbb{R}^3} e^{-\sigma(t-r)}\sigma'(t-r)r^{p-2}u^2\,dxdt$$

to the left side of (2.19) to obtain (2.17).

To prove (2.18), we start with the identity

$$(\log r)^{-2}e^{-\sigma(t-r)}\sigma'(t-r)r^{-1}=-(\partial_t+\partial_r)\left((\log r)^{-1}e^{-\sigma(t-r)}\sigma'(t-r)\right).$$

From here, we integrate by parts in $(\partial_t+\partial_r)$ so that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^3}\beta(r)(\log r)^{-2}e^{-\sigma(t-r)}\sigma'(t-r)r^{-1}u^2\,dxdt&\\ =2\int_0^T\int_{\mathbb{R}^3}\beta(r)(\log r)^{-1}e^{-\sigma(t-r)}\sigma'(t-r)r^{-1}(\partial_t+\partial_r)(ru)\cdot u\,dxdt&\\ +\int_0^T\int_{\mathbb{R}^3}\beta'(r)(\log r)^{-1}e^{-\sigma(t-r)}\sigma'(t-r)u^2\,dxdt-\int_{\mathbb{R}^3}\beta(r)(\log r)^{-1}e^{-\sigma(t-r)}\sigma'(t-r)u^2\,dx\Big|_0^T&.\end{aligned} \tag{2.20}$$

Applying the Schwarz inequality to the first term on the right side of (2.20) gives

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^3}\beta(r)(\log r)^{-1}e^{-\sigma(t-r)}\sigma'(t-r)r^{-1}(\partial_t+\partial_r)(ru)\cdot u\,dxdt&\\ \le\left(\int_0^T\int_{\mathbb{R}^3}\beta(r)e^{-\sigma(t-r)}\sigma'(t-r)r^{-1}[(\partial_t+\partial_r)(ru)]^2\,dxdt\right)^{1/2}&\\ \cdot\left(\int_0^T\int_{\mathbb{R}^3}\beta(r)(\log r)^{-2}e^{-\sigma(t-r)}\sigma'(t-r)r^{-1}u^2\,dxdt\right)^{1/2}&.\end{aligned}$$

Then the second factor above on the right may be bootstrapped to the left side of (2.20) after moving the time boundary piece at $t=T$ to the left side.

For the second term on the right of (2.20), we note that $\beta'(r)$ is supported on $[2,4]$. As this set is compact, we can bound this term using $\|u\|_{LE^1}^2$. Therefore, we have

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^3}\beta(r)(\log r)^{-2}e^{\sigma(t-r)}\sigma'(t-r)r^{-1}u^2\,dxdt+\int_{\mathbb{R}^3}\beta(r)(\log r)^{-1}e^{-\sigma(T-r)}\sigma'(T-r)u(T,x)^2\,dx&\\ \lesssim\int_0^T\int_{\mathbb{R}^3}e^{-\sigma(t-r)}\sigma'(t-r)r^{-1}[(\partial_t+\partial_r)(ru)]^2\,dxdt&\\ +\int_{\mathbb{R}^3}\beta(r)(\log r)^{-1}e^{-\sigma(-r)}\sigma'(-r)u(0,x)^2\,dx+\|u\|_{LE^1}^2&.\end{aligned} \tag{2.21}$$

From here, we use $\langle r\rangle\approx r$ on the support of $\beta(r)$ so that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^3}(\log\langle r\rangle)^{-2}e^{-\sigma(t-r)}\sigma'(t-r)r^{-1}u^2\,dxdt\lesssim\int_0^T\int_{\mathbb{R}^3}\beta(r)(\log r)^{-2}e^{-\sigma(t-r)}\sigma'(t-r)r^{-1}u^2\,dxdt&\\ +\int_0^T\int_{\mathbb{R}^3}(1-\beta(r))(\log\langle r\rangle)^{-2}e^{-\sigma(t-r)}\sigma'(t-r)r^{-1}u^2\,dxdt&.\end{aligned} \tag{2.22}$$

For the first term on the right of (2.22), we apply (2.21). Since $1-\beta(r)$ is compactly supported, the second term is controlled using $\|u\|^2_{LE^1}$. This completes the proof of (2.18). $\square$

The following lemma provides our second Hardy-type inequality for the proof of Proposition 2.1.

Lemma 2.5. *Suppose $T > 0$, $u \in C^2([0,T]\times\mathbb{R}^3)$ and $\sigma \in C^1(\mathbb{R})$. Assume u has compactly supported initial data and σ is nondecreasing. Then, for each fixed constant $0 < p < 2$,*

$$\int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)} r^{p-3} u^2 \, dxdt + \sup_{t\in[0,T]} \int_{\mathbb{R}^3} e^{-\sigma(t-r)} r^{p-2} u(t,x)^2 \, dx \\ \lesssim \int_{\mathbb{R}^3} e^{-\sigma(-r)} r^{p-2} u(0,x)^2 \, dx + \int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)} r^{p-3} [(\partial_t + \partial_r)(ru)]^2 \, dxdt. \quad (2.23)$$

Proof. We first write

$$(2-p) e^{-\sigma(t-r)} r^{p-3} = -(\partial_t + \partial_r)\left(e^{-\sigma(t-r)} r^{p-2}\right)$$

and use integration by parts so that

$$\begin{aligned}(2-p)&\int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)} r^{p-3} u^2 \, dxdt \\ &= 2\int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)} r^{p-3} u(\partial_t + \partial_r)(ru) \, dxdt - \int_{\mathbb{R}^3} e^{-\sigma(t-r)} r^{p-2} u^2 \, dx\Big|_0^T \\ &\le 2\left(\int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)} r^{p-3} u^2 \, dxdt\right)^{1/2} \cdot \left(\int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)} r^{p-3} [(\partial_t + \partial_r)(ru)]^2 \, dxdt\right)^{1/2} \\ &\qquad - \int_{\mathbb{R}^3} e^{-\sigma(t-r)} r^{p-2} u^2 \, dx\Big|_0^T\end{aligned} \quad (2.24)$$

where the inequality follows by applying the Schwarz inequality. To establish (2.23), since $p < 2$, the term $\int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)} r^{p-3} u^2 \, dxdt$ can be bootstrapped to the left-hand side of (2.24). $\square$

Now we will prove Proposition 2.1 using Lemmas 2.3, 2.4, and 2.5.

Proof of Proposition 2.1. In addition to the lemmas above, we will also use the following estimate:

$$\int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)} r^{p-1} [(\partial_t + \partial_r)u]^2 \, dxdt \\ \lesssim \int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)} r^{p-3} [(\partial_t + \partial_r)(ru)]^2 \, dxdt + \int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)} r^{p-3} u^2 \, dxdt. \quad (2.25)$$

This equation follows directly from the identity

$$(\partial_t + \partial_r)u = r^{-1}(\partial_t + \partial_r)(ru) - r^{-1}u.$$

Now, we may use (2.25) along with Lemmas 2.3, 2.4, and 2.5 with $0 < p < 1$ to conclude

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)}\sigma'(t-r)r^{p-2}u^2 dxdt + \int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)}r^{p-3}u^2 dxdt + \int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)}r^{p-1}|\nabla\!\!\!/ u|^2 dxdt \\
&+ \int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)}\sigma'(t-r)r^{p-2}[(\partial_t+\partial_r)(ru)]^2\, dxdt + \int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)}r^{p-1}[(\partial_t+\partial_r)u]^2\, dxdt \\
&\quad + \sup_{t\in[0,T]} \Big(\int_{\mathbb{R}^3} [(\partial_t+\partial_r)(ru(t,x))]^2 e^{-\sigma(t-r)}r^{p-2}\, dx + \int_{\mathbb{R}^3} |\nabla\!\!\!/ u(t,x)|^2 e^{-\sigma(t-r)}r^p\, dx \\
&\quad\quad + \int_{\mathbb{R}^3} e^{-\sigma(t-r)}\sigma'(t-r)r^{p-1}u(t,x)^2\, dx + \int_{\mathbb{R}^3} e^{-\sigma(t-r)}r^{p-2}u(t,x)^2\, dx \Big) \\
&\quad\quad \lesssim \int_{\mathbb{R}^3} [(\partial_t+\partial_r)(ru(0,x))]^2 e^{-\sigma(-r)}r^{p-2}\, dx + \int_{\mathbb{R}^3} |\nabla\!\!\!/ u(0,x)|^2 e^{-\sigma(-r)}r^p\, dx \\
&\quad\quad\quad + \int_{\mathbb{R}^3} e^{-\sigma(-r)}\sigma'(-r)r^{p-1}u(0,x)^2\, dx + \int_{\mathbb{R}^3} e^{-\sigma(-r)}r^{p-2}u(0,x)^2\, dx \\
&\quad\quad\quad\quad + \left| \int_0^T \int_{\mathbb{R}^3} \Box u e^{-\sigma(t-r)}r^p \left(\partial_t + \partial_r + \frac{1}{r}\right) u\, dxdt \right|.
\end{aligned}$$

Similarly, if $p = 1$,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} (\log\langle r\rangle)^{-2} e^{-\sigma(t-r)}\sigma'(t-r)r^{-1}u^2\, dxdt + \int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)}r^{-2}u^2 dxdt \\
&\quad + \int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)}|\nabla\!\!\!/ u|^2 dxdt + \int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)}[(\partial_t+\partial_r)u]^2 dxdt \\
&\quad\quad + \int_0^T \int_{\mathbb{R}^3} e^{-\sigma(t-r)}\sigma'(t-r)r^{-1}[(\partial_t+\partial_r)(ru)]^2 dxdt \\
&\sup_{t\in[0,T]} \Big(\int_{\mathbb{R}^3} e^{-\sigma(t-r)}r^{-1}[(\partial_t+\partial_r)(ru(t,x))]^2\, dx + \int_{\mathbb{R}^3} e^{-\sigma(t-r)}r|\nabla\!\!\!/ u(t,x)|^2\, dx \\
&\quad + \int_{\mathbb{R}^3} (\log\langle r\rangle)^{-1} e^{-\sigma(t-r)}\sigma'(t-r)u(t,x)^2\, dx + \int_{\mathbb{R}^3} e^{-\sigma(t-r)}r^{-1}u(t,x)^2\, dx \Big) \\
&\quad \lesssim \int_{\mathbb{R}^3} e^{-\sigma(-r)}r^{-1}[(\partial_t+\partial_r)(ru(0,x))]^2\, dx + \int_{\mathbb{R}^3} e^{-\sigma(-r)}r|\nabla\!\!\!/ u(0,x)|^2\, dx \\
&\quad\quad + \int_{\mathbb{R}^3} (\log\langle r\rangle)^{-1} e^{-\sigma(-r)}\sigma'(-r)u(0,x)^2\, dx + \int_{\mathbb{R}^3} e^{-\sigma(-r)}r^{-1}u(0,x)^2\, dx \\
&\quad\quad\quad + \|u\|^2_{LE^1} + \left| \int_0^T \int_{\mathbb{R}^3} \Box u e^{-\sigma(t-r)}r \left(\partial_t + \partial_r + \frac{1}{r}\right) u\, dxdt \right|.
\end{aligned}$$

$\square$

CHAPTER 3

Estimates on Dyadic Regions

As the local energy estimate (2.5) is well-suited to be applied to the localized regions C_τ^R and C_τ^U, it will be useful to establish pointwise bounds on these regions. In this chapter, we first establish various derivative bounds that will be convenient in the proof of Theorem 1.1. Subsequently, we record a Klainerman-Sobolev type estimate from Metcalfe, Tataru, and Tohaneanu [16]. This $L^\infty L^\infty - L^2L^2$ estimate will be our main source of decay for the proof of Theorem 1.1.

3.1 Derivative Bounds

In this section, we will establish bounds for ∂w and $\not\partial\partial w$ to use in the proof of Theorem 1.1. The following results primarily rely on the decomposition of the space-time gradient (1.7) and the scaling vector field identity (1.8). We further note that the following results will not be used on $C_T^{U=1}$ as there is no lower bound on $|t-r|$ on these regions. In the proof of Theorem 1.1, we will deal with the case when $U = 1$ separately.

Since the local energy estimate (2.5) only involves good derivatives, $\not\partial$, we begin with a lemma relating bad derivatives of the form $\partial_t - \partial_r$ to terms involving either a good derivative or additional decay at the expense of a vector field.

Lemma 3.1. *Assume $T \geq 1$, $1 \leq R \leq T/4$, $1 < U \leq T/4$ and $w \in C^1([7T/8, 17T/8] \times \mathbb{R}^3)$. Then for $(t,x) \in \tilde{C}_T^R$,*

$$|(\partial_t - \partial_r)w(t,x)| \lesssim T^{-1}|Sw(t,x)| + |(\partial_t + \partial_r)w(t,x)| \tag{3.1}$$

and for $(t,x) \in \tilde{C}_T^U$,

$$|(\partial_t - \partial_r)w(t,x)| \lesssim U^{-1}|Sw(t,x)| + U^{-1}T|(\partial_t + \partial_r)w(t,x)|. \tag{3.2}$$

Proof. This result simply follows from (1.8) and the triangle inequality. Additionally, we use $t+r \lesssim T$ and $|t-r| \gtrsim T$ on $\tilde{C}_T^R$, and $|t-r| \approx U$ and $t+r \approx T$ on $\tilde{C}_T^U$ when $1 < U \leq T/4$. □

Now we present bounds for the full space-time gradient ∂w in terms of good derivatives and terms involving additional decay at the expense of a vector field.

Proposition 3.2. *Assume $T \geq 1$, $1 \leq R \leq T/4$, $1 < U \leq T/4$, and $w \in C^1([7T/8, 17T/8] \times \mathbb{R}^3)$. Then for $(t,x) \in \tilde{C}_T^R$,*

$$|\partial w(t,x)| \lesssim |\bar{\partial} w(t,x)| + R^{-1}|Sw(t,x)|, \tag{3.3}$$

and for $(t,x) \in \tilde{C}_T^U$,

$$|\partial w(t,x)| \lesssim |\bar{\partial} w(t,x)| + U^{-1}|Sw(t,x)| + U^{-1}|(\partial_t + \partial_r)(rw(t,x))|. \tag{3.4}$$

Proof. These estimates follow directly from (1.7) and an application of Lemma 3.1 to terms involving $(\partial_t - \partial_r)$. $\square$

Next, we establish a bound for terms involving a derivative paired with a good derivative. Whenever both derivatives are of the form $\partial_t + \partial_r$, it will be convenient to bound one such derivative by terms involving either $\partial_t - \partial_r$ or additional decay with an extra vector field. The bound on $\partial_t + \partial_r$ is contained in the following lemma. The proof is nearly identical to that of Lemma 3.1. Here we also use $t + r \gtrsim R$ on $\tilde{C}_T^R$.

Lemma 3.3. *Assume $T \geq 1$, $1 \leq R \leq T/4$, $1 < U \leq T/4$, and $w \in C^1\left([7T/8, 17T/8] \times \mathbb{R}^3\right)$. Then, for $(t,x) \in \tilde{C}_T^R$*

$$|(\partial_t + \partial_r)w(t,x)| \lesssim R^{-1}|Sw(t,x)| + |(\partial_t - \partial_r)w(t,x)|. \tag{3.5}$$

Similarly, for $(t,x) \in \tilde{C}_T^U$,

$$|(\partial_t + \partial_r)w(t,x)| \lesssim T^{-1}|Sw(t,x)| + UT^{-1}|(\partial_t - \partial_r)w(t,x)|. \tag{3.6}$$

Now we state and prove our estimate for terms of the form $|\bar{\partial}\partial w|$.

Proposition 3.4. *Assume $T \geq 1$, $1 \leq R \leq T/4$, $1 < U \leq T/4$ and $w \in C^2\left([7T/8, 17T/8] \times \mathbb{R}^3\right)$. Then for $(t,x) \in \tilde{C}_T^R$,*

$$|\bar{\partial}\partial w| \lesssim R^{-1}|\partial \Gamma^{\leq 1} w| + |\Box w|, \tag{3.7}$$

and for $(t,x) \in \tilde{C}_T^U$,

$$|\bar{\partial}\partial w| \lesssim T^{-1}|\partial\Gamma^{\leq 1} w| + |\Box w|. \tag{3.8}$$

Proof. We begin by writing

$$|\bar{\partial}\partial w| \leq |(\partial_t + \partial_r)\partial w| + |\nabla\!\!\!/\,\partial w|. \tag{3.9}$$

From here, we will look at each term on the right of (3.9) individually.

For the first term, we apply (1.18), (1.19), and (1.7) so that for $(t,x) \in \tilde{C}_T^R$,

$$|(\partial_t + \partial_r)\partial w| \lesssim |(\partial_t + \partial_r)^2 w| + |(\partial_t^2 - \partial_r^2)w| + |\nabla\!\!\!/\,(\partial_t + \partial_r)w| + R^{-1}|\nabla\!\!\!/\, w|,$$

and for $(t,x) \in \tilde{C}_T^U$,

$$|(\partial_t + \partial_r)\partial w| \lesssim |(\partial_t + \partial_r)^2 w| + |(\partial_t^2 - \partial_r^2)w| + |\nabla\!\!\!/\,(\partial_t + \partial_r)w| + T^{-1}|\nabla\!\!\!/\, w|.$$

To handle terms of the form $|(\partial_t + \partial_r)^2 w|$, we apply Lemma 3.3 and (1.15) so that on $\tilde{C}_T^R$,

$$|(\partial_t + \partial_r)^2 w| \lesssim R^{-1}|\bar{\partial}\Gamma^{\leq 1} w| + |(\partial_t^2 - \partial_r^2)w|,$$

and on $\tilde{C}_T^U$,

$$|(\partial_t + \partial_r)^2 w| \lesssim T^{-1}|\bar{\partial}\Gamma^{\leq 1} w| + |(\partial_t^2 - \partial_r^2)w|.$$

Then using the identity

$$\partial_t^2 - \partial_r^2 = \Box + \nabla\!\!\!/ \cdot \nabla\!\!\!/ + \frac{2}{r}\partial_r$$

along with (1.9) and (1.16), it follows that for $(t,x) \in \tilde{C}_T^R$,

$$|(\partial_t^2 - \partial_r^2)w| \lesssim |\Box w| + R^{-1}|\bar{\partial}\Gamma^{\leq 1} w| + R^{-1}|\partial w|,$$

and for $(t,x) \in \tilde{C}_T^U$,

$$|(\partial_t^2 - \partial_r^2)w| \lesssim |\Box w| + T^{-1}|\bar{\partial}\Gamma^{\leq 1} w| + T^{-1}|\partial w|.$$

Finally, (1.9) and (1.17) can be used to show

$$|\nabla\!\!\!/(\partial_t + \partial_r)w| \lesssim R^{-1}|\partial\!\!\!/\Gamma^{\leq 1}w|$$

on $\tilde{C}_T^R$, and

$$|\nabla\!\!\!/(\partial_t + \partial_r)w| \lesssim T^{-1}|\partial\!\!\!/\Gamma^{\leq 1}w|$$

on $\tilde{C}_T^U$, which completes the desired result for the first term on the right of (3.9).

To control the second term on the right of (3.9), we use (1.9), (1.16), and (1.17). On $\tilde{C}_T^R$, this yields

$$|\nabla\!\!\!/\partial w| \lesssim R^{-1}|\partial\Gamma^{\leq 1}w|,$$

and on $\tilde{C}_T^U$, we find

$$|\nabla\!\!\!/\partial w| \lesssim T^{-1}|\partial\Gamma^{\leq 1}w|.$$

This completes the proofs of (3.7) and (3.8). □

3.2 Sobolev Estimate

To obtain pointwise bounds from L^2L^2 bounds, we will apply Sobolev embeddings on each of the double-dyadic regions C_τ^R and C_τ^U after making an appropriate change of coordinates. The result, which originates from the work of Metcalfe, Tataru, and Tohaneanu [16], is recorded in the following proposition.

Proposition 3.5. *For $T \geq 1$, $1 \leq R \leq T/4$, $1 \leq U \leq T/4$, and $w \in C^3\left(\left[\frac{7}{8}T, \frac{17}{8}T\right] \times \mathbb{R}^3\right)$,*

$$\|w\|_{L^\infty L^\infty(C_T^R)} \lesssim T^{-1/2}R^{-3/2} \sum_{j+|\alpha|\leq 2} \|S^j\Omega^\alpha w\|_{L^2L^2(\tilde{C}_T^R)} + T^{-1/2}R^{-1/2} \sum_{j+|\alpha|\leq 2} \|S^j\Omega^\alpha \partial w\|_{L^2L^2(\tilde{C}_T^R)} \tag{3.10}$$

and

$$\|w\|_{L^\infty L^\infty(C_T^U)} \lesssim T^{-3/2}U^{-1/2} \sum_{j+|\alpha|\leq 3} \|S^j\Omega^\alpha w\|_{L^2L^2(\tilde{C}_T^U)} + T^{-3/2}U^{1/2} \sum_{j+|\alpha|\leq 2} \|S^j\Omega^\alpha \partial w\|_{L^2L^2(\tilde{C}_T^U)}. \tag{3.11}$$

Proof. We begin by fixing a cutoff function $\beta \in C^\infty(\mathbb{R})$ so that $\beta \equiv 1$ on $[1, 2]$ and the support of β is contained in $[7/8, 17/8]$. For any $R > 1$ and $T \geq 1$, we define $\beta_R(r) = \beta(\frac{r}{R})$ and $\beta_T(t) = \beta(\frac{t}{T})$.

When $R = 1$, we instead use a cutoff $\beta_{R=1}$ so that $\beta_{R=1} \equiv 1$ on $[0, 2]$ with support in $[0, 17/8]$.

To establish (3.10), consider

$$\|w\|_{L^\infty L^\infty(C_T^R)} \leq \|\beta_R \beta_T w\|_{L^\infty L^\infty}. \tag{3.12}$$

From here, we will convert to the exponential coordinates (s, ρ, θ, ϕ) where $t = e^s$, $r = e^{s+\rho}$, and θ and ϕ are the usual coordinates on $\mathbb{S}^2$. That is, θ and ϕ satisfy

$$\begin{aligned} x_1 &= r \sin\phi \cos\theta \\ x_2 &= r \sin\phi \sin\theta \\ x_3 &= r \cos\phi. \end{aligned}$$

Define the function v so that $v(s, \rho, \theta, \phi) = w(e^s, e^{s+\rho} \sin\phi \cos\theta, e^{s+\rho} \sin\phi \sin\theta, e^{s+\rho} \cos\phi)$.

Next, we apply a Sobolev embedding in s, θ and ϕ so that

$$|\beta_R(e^{s+\rho})\beta_T(e^s)v(s, \rho, \theta, \phi)|^2 \leq \|\partial^{\leq 2}_{s,\theta,\phi}(\beta_R \beta_T v)(\cdot, \rho, \cdot, \cdot)\|^2_{L^2_s L^2_\theta L^2_\phi} \tag{3.13}$$

where derivatives are with respect to s, θ, and ϕ. Subsequently, we use the Fundamental Theorem of Calculus together with (3.12) and (3.13) so that

$$\|w\|^2_{L^2 L^2(C_T^R)} \leq \iiiint \partial_\rho \left(|\partial^{\leq 2}(\beta_R \beta_T v(s, \rho, \theta, \phi))|^2\right) ds d\rho d\theta d\phi. \tag{3.14}$$

We compute the following derivatives:

$$\partial_s v = r\partial_r w + t\partial_t w = Sw$$

$$\partial_\rho v = r\partial_r w$$

$$\partial_\theta v = -\Omega_{12} w$$

$$\partial_\phi v = \cos\theta \Omega_{13} w + \sin\theta \Omega_{23} w$$

By combining these with (3.14), converting to Euclidean coordinates (t, x), and applying the Schwarz inequality, we obtain

$$\|w\|^2_{L^2L^2(C_T^R)} \lesssim T^{-1}R^{-3}\sum_{j+|\alpha|\le 2}\|S^j\Omega^\alpha w\|^2_{L^2L^2(\tilde{C}_T^R)} + T^{-1}R^{-2}\left(\sum_{j+|\alpha|\le 2}\|S^j\Omega^\alpha w\|_{L^2L^2(\tilde{C}_T^R)}\right)\cdot\left(\sum_{j+|\alpha|\le 2}\|S^j\Omega^\alpha \partial_r w\|_{L^2L^2(\tilde{C}_T^R)}\right). \quad (3.15)$$

Here a decay factor of $T^{-1}R^{-3}$ comes from the change of coordinates and the enlargements are due to the tails of the cutoff functions. (3.10) follows immediately from (3.15).

For the proof of (3.11), in the case where $U = 1$, we may choose R sufficiently close to T so that $C_T^{U=1} \subseteq C_T^R$ and apply (3.15). Treating ∂_r as an additional vector field gives the desired result.

For $U > 1$, we instead use cutoffs $\beta_U(t-r) = \beta(\frac{|t-r|}{U})$ and $\beta_T(t) = \beta(\frac{t}{T})$ for $T \ge 1$. Additionally, we use the exponential coordinates (s, ρ, θ, ϕ) where $t = e^s$ and $t - r = e^{s+\rho}$. θ and ϕ are as before.

We use similar arguments to that above. We will again apply a Sobolev embedding in s, θ, and ϕ and subsequently use the Fundamental Theorem of Calculus so that

$$\|w\|^2_{L^2L^2(C_T^U)} \le \iiiint \partial_\rho\left(|\partial^{\le 2}(\beta_U\beta_T v(s,\rho,\theta,\phi))|^2\right) ds d\rho d\theta d\phi.$$

Again, we compute $\partial_s v = Sw$, but now $\partial_\rho v = (t-r)\partial_r w$. Derivatives in θ and ϕ are as above. Therefore, by returning to Euclidean coordinates and applying the Schwarz inequality, we find

$$\|w\|^2_{L^2L^2(C_T^U)} \lesssim T^{-3}U^{-1}\left(\sum_{j+|\alpha|\le 2}\|S^j\Omega^\alpha w\|^2_{L^2L^2(\tilde{C}_T^U)}\right) + T^{-3}\left(\sum_{j+|\alpha|\le 2}\|S^j\Omega^\alpha w\|_{L^2L^2(\tilde{C}_T^U)}\right)\cdot\left(\sum_{j+|\alpha|\le 2}\|S^j\Omega^\alpha \partial_r w\|_{L^2L^2(\tilde{C}_T^U)}\right).$$

This directly implies (3.11).

□

In the proof of Theorem 1.1, we will not care which vector fields appear and it will be beneficial to identify derivatives of the form $(\partial_t + \partial_r)$. Therefore, the following corollary is the version of Proposition 3.5 that we will use in Chapter 4. We also establish the desired result when $U = 1$.

Corollary 3.6. *For $T \geq 1$, $1 \leq R \leq T/4$, $1 \leq U \leq T/4$, and $w \in C^3\left(\left[\frac{7}{8}T, \frac{17}{8}T\right] \times \mathbb{R}^3\right)$,*

$$\|w\|_{L^\infty L^\infty(C_T^R)} \lesssim T^{-1/2}R^{-3/2}\|\Gamma^{\leq 3}w\|_{L^2L^2(\tilde{C}_T^R)} + T^{-1/2}R^{-1/2}\|(\partial_t + \partial_r)\Gamma^{\leq 2}w\|_{L^2L^2(\tilde{C}_T^R)} \tag{3.16}$$

and

$$\|w\|_{L^\infty L^\infty(C_T^U)} \lesssim T^{-3/2}U^{-1/2}\|\Gamma^{\leq 3}w\|_{L^2L^2(\tilde{C}_T^U)} + T^{-3/2}U^{-1/2}\|(\partial_t + \partial_r)(r\Gamma^{\leq 2}w)\|_{L^2L^2(\tilde{C}_T^U)}. \tag{3.17}$$

Proof. We begin by applying Proposition 3.5. For the second terms on the right hands sides of (3.10) and (3.11), we use (1.12), (1.13), and Proposition 3.2 for $R \geq 1$ and $U > 1$ so that

$$\begin{aligned} T^{-1/2}R^{-1/2} \sum_{j+|\alpha|\leq 2} \|S^j\Omega^\alpha \partial w\|_{L^2L^2(\tilde{C}_T^R)} &\lesssim T^{-1/2}R^{-1/2}\|\bar{\nabla}\Gamma^{\leq 2}w\|_{L^2L^2(\tilde{C}_T^R)} \\ &+ T^{-1/2}R^{-1/2}\|(\partial_t + \partial_r)\Gamma^{\leq 2}w\|_{L^2L^2(\tilde{C}_T^R)} + T^{-1/2}R^{-3/2}\|\Gamma^{\leq 3}w\|_{L^2L^2(\tilde{C}_T^R)} \end{aligned}$$

and

$$\begin{aligned} T^{-3/2}U^{1/2} \sum_{j+|\alpha|\leq 2} \|S^j\Omega^\alpha \partial w\|_{L^2L^2(\tilde{C}_T^U)} &\lesssim T^{-3/2}U^{1/2}\|\bar{\nabla}\Gamma^{\leq 2}w\|_{L^2L^2(\tilde{C}_T^U)} \\ &+ T^{-3/2}U^{1/2}\|(\partial_t + \partial_r)\Gamma^{\leq 2}w\|_{L^2L^2(\tilde{C}_T^U)} + T^{-3/2}U^{-1/2}\|\Gamma^{\leq 3}w\|_{L^2L^2(\tilde{C}_T^U)} \\ &+ T^{-3/2}U^{-1/2}\|(\partial_t + \partial_r)(rw)\|_{L^2L^2(\tilde{C}_T^U)}. \end{aligned}$$

The desired result follows from an application of (1.9) and $T^{-1} \lesssim U^{-1}$ on $\tilde{C}_T^U$ for $1 < U \leq T/4$.

In the case where $U = 1$, we may choose $\hat{R}$ sufficiently close to T so that $C_T^{U=1} \subseteq C_T^{\hat{R}}$. The desired result follows from applying (3.15) over $C_T^{\hat{R}}$ where we simply treat ∂_r as a vector field and use the fact that $T \approx \hat{R}$.

$\square$

CHAPTER 4

Proof of Main Theorem

In this chapter, we present the proof of Theorem 1.1. To do so, we will solve (1.1) using an iteration method. Generally, one defines a sequence of functions u_k so that $u_0 \equiv 0$ and for $k \geq 1$, u_k solves

$$\begin{cases} \Box u_k^I = Q^I(u_{k-1}, u_{k-1}') \\ u_k^I(0, \cdot) = f^I \\ \partial_t u_k^I(0, \cdot) = g^I. \end{cases}$$

One must show this sequence of functions converges. Then the limit of the sequence is the desired solution.

Under the simplifying assumption (1.4), we take u_k to solve

$$\begin{cases} \Box u_k^I = a_{JK}^{I,\alpha} u_{k-1}^J \partial_\alpha u_{k-1}^K + b_{JK}^{I,\alpha\beta} \partial_\alpha u_{k-1}^J \partial_\beta u_{k-1}^K \\ u_k^I(0, \cdot) = f^I \\ \partial_t u_k^I(0, \cdot) = g^I. \end{cases} \tag{4.1}$$

with $f, g \in (C^\infty(\mathbb{R}^3))^M$ satisfying (1.5). Although this is not nearly sharp, we will work with $N = 60$ throughout. To show that u_k converges, we will first show that u_k is bounded. This boundedness will then be used to prove that u_k is Cauchy. As the spaces involved are complete, this suffices to establish convergence.

4.1 Proof of Boundedness

In this section, we will use an induction argument to establish a certain boundedness for the sequence of functions u_k. In particular, we will create a bound for a sum of appropriately weighted norms of u_k.

To begin, for a fixed $T \le T_\varepsilon$, we define

$$\begin{aligned} M_k = \|\Gamma^{\le 60} u_k\|_{LE^1} + \|\partial\Gamma^{\le 60} u_k\|_{L^\infty L^2} + \|\langle r\rangle^{1/2}\bar\partial\Gamma^{\le 60} u\|_{L^\infty L^2} + \|r^{-1/2}\Gamma^{\le 60} u\|_{L^\infty L^2} + \|\bar\partial\Gamma^{\le 60} u\|_{L^2L^2} \\ + \|r^{-1}\Gamma^{\le 60} u\|_{L^2L^2} + \sup_{U\ge 1} \|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\le 60}u)\|_{L^2L^2(X_U)} \\ + \sup_{U\ge 1}\|r^{-1/2}(\log\langle r\rangle)^{-1}\langle t-r\rangle^{-1/2}\Gamma^{\le 60}u\|_{L^2L^2(X_U)} \\ + \left(\sum_{\tau\le T}\sum_{R\le \tau/4}\|\partial\Gamma^{\le 50}u_k\|^2_{L^2L^2(\tilde C^R_\tau)}\right)^{1/2} + \left[\sum_{\tau\le T}\sum_{R\le\tau/4}\Big(R\|\bar\partial\partial\Gamma^{\le 40}u_k\|_{L^2L^2(\tilde C^R_\tau)}\Big)^2\right]^{1/2} \\ + \sup_{U\ge 1}\left[\sum_{\tau\ge 4U}\left(\frac{U^{1/2}}{\tau^{1/2}\log\langle\tau\rangle}\|\partial\Gamma^{\le 50}u_k\|_{L^2L^2(\tilde C^U_\tau)}\right)^2\right]^{1/2} \\ + \sup_{U\ge 1}\left[\sum_{\tau\ge 4U}\left(\frac{\tau^{1/2}U^{1/2}}{\log\langle\tau\rangle}\|\bar\partial\partial\Gamma^{\le 40}u_k\|_{L^2L^2(\tilde C^U_\tau)}\right)^2\right]^{1/2}. \end{aligned} \tag{4.2}$$

The terms of M_k will be enumerated as I_k, II_k, $\dots$, XII_k so that

$$M_k = I_k + II_k + III_k + IV_k + V_k + VI_k + VII_k + VIII_k + IX_k + X_k + XI_k + XII_k.$$

We will use induction to show that for each $k \ge 0$,

$$M_k \le 2C_0\varepsilon \tag{4.3}$$

for $T \le T_\varepsilon$ where C_0 is a constant independent of k. To do so, we will first show for a constant $C > 0$ that

$$M_k^2 \le (C_0\varepsilon)^2 + C(\log\langle T\rangle)^3 M_{k-1}^2 M_k + C(\log\langle T\rangle)^5 M_{k-1}^4. \tag{4.4}$$

Since $u_0 \equiv 0$, $M_1 \le C_0\varepsilon$ immediately follows from (4.4), which establishes the base case. For induction, we will assume for $k \ge 1$, that $M_{k-1} \le 2C_0\varepsilon$. Using (4.4) and (1.6), for c and ε sufficiently small, (4.3) follows.

4.1.1 Terms I_k and II_k

We will apply the standard local energy estimate (2.1) in order to show

$$I_k^2 + II_k^2 \leq (C_1\varepsilon)^2 + C(\log\langle T\rangle)^{5/2} M_{k-1}^2 M_k. \tag{4.5}$$

Using (2.1), we see

$$\begin{aligned}\|\Gamma^{\leq 60} u_k\|_{LE^1}^2 + \|\partial\Gamma^{\leq 60} u_k\|_{L^\infty L^2}^2 &\lesssim \|\partial\Gamma^{\leq 60} u_k(0,\cdot)\|_{L^2}^2 \\ &\quad + \int_0^T \int |\Box\Gamma^{\leq 60} u_k| \left(|\partial\Gamma^{\leq 60} u_k| + \frac{|\Gamma^{\leq 60} u_k|}{r}\right) dx dt.\end{aligned} \tag{4.6}$$

The first term on the right of (4.6) is bounded by $(C_1\varepsilon)^2$ using (1.5) for some constant $C_1 > 0$.

For the second term on the right of (4.6), we will use (1.10). By the product rule, we note that on any given term there will be one factor with at most half of the 60 vector fields. As a result, we have

$$|\Box\Gamma^{\leq 60} u_k| \lesssim |\Gamma^{\leq 30} u_{k-1}||\partial\Gamma^{\leq 60} u_{k-1}| + |\partial\Gamma^{\leq 30} u_{k-1}||\Gamma^{\leq 60} u_{k-1}| + |\partial\Gamma^{\leq 30} u_{k-1}||\partial\Gamma^{\leq 60} u_{k-1}|. \tag{4.7}$$

For each term in (4.7), as in (1.20), we will bound the integrals over C_τ^R and C_τ^U and sum over $R \leq \tau/4$, $U \leq \tau/4$, and $\tau \leq T$.

Using (3.16), (3.17), and the Schwarz inequality on the first term from (4.7), we see

$$\begin{aligned}&\int\int_{C_\tau^R} |\Gamma^{\leq 30} u_{k-1}||\partial\Gamma^{\leq 60} u_{k-1}| \left(|\partial\Gamma^{\leq 60} u_k| + \frac{|\Gamma^{\leq 60} u_k|}{r}\right) dx dt \\ &\quad \lesssim \left(\|r^{-1}\Gamma^{\leq 33} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + \|\bar{\partial}\Gamma^{\leq 32} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\right) \|\langle r\rangle^{-1/2}\partial\Gamma^{\leq 60} u_{k-1}\|_{L^2L^2(C_\tau^R)} \\ &\qquad \times \left(\|\langle r\rangle^{-1/2}\partial\Gamma^{\leq 60} u_k\|_{L^2L^2(C_\tau^R)} + \|\langle r\rangle^{-1/2} r^{-1}\Gamma^{\leq 60} u_k\|_{L^2L^2(C_\tau^R)}\right)\end{aligned}$$

and

$$\int\int_{C_\tau^U} |\Gamma^{\le 30}u_{k-1}||\partial\Gamma^{\le 60}u_{k-1}|\left(|\partial\Gamma^{\le 60}u_k| + \frac{|\Gamma^{\le 60}u_k|}{r}\right)dxdt$$
$$\lesssim \left(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\le 33}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\le 32}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\right)$$
$$\times \|\langle r\rangle^{-1/2}\partial\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}\left(\|\langle r\rangle^{-1/2}\partial\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^U)} + \|\langle r\rangle^{-1/2}r^{-1}\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^U)}\right).$$

Since terms IX_k and XII_k allow for at most 50 vector fields, we may use the LE^1 norm to control terms of the form $\|\partial\Gamma^{\le A}u\|_{L^2L^2}$ where $A > 50$ when paired with an appropriate weight. When a full decay factor $\langle r\rangle^{-1}$ is available, we may bound

$$\|\langle r\rangle^{-1}\partial\Gamma^{\le A}u\|^2_{L^2L^2} = \sum_{j\ge 0} 2^{-j}\left(2^{-j}\|\partial\Gamma^{\le A}u\|^2_{L^2L^2([0,T]\times A_{2^j})}\right) \lesssim \|\Gamma^{\le A}u\|^2_{LE^1}. \tag{4.8}$$

When the available decay is only $\langle r\rangle^{-1/2}$, this bound comes with a logarithmic loss. That is,

$$\|\langle r\rangle^{-1/2}\partial\Gamma^{\le A}u\|^2_{L^2L^2} \le \sum_{0\le j\lesssim \log\langle T\rangle} 2^{-j}\|\partial\Gamma^{\le A}u\|^2_{L^2L^2([0,T]\times A_{2^j})} \lesssim \log\langle T\rangle\|\Gamma^{\le A}u\|^2_{LE^1} \tag{4.9}$$

where we also use the fact that the initial data is supported where $\{|x| \le 1\}$. Therefore, after summing and applying (4.9), we conclude

$$\int_0^T\int |\Gamma^{\le 30}u_{k-1}||\partial\Gamma^{\le 60}u_{k-1}|\left(|\partial\Gamma^{\le 60}u_k| + \frac{|\Gamma^{\le 60}u_k|}{r}\right)dxdt$$
$$\lesssim (VI_{k-1}+V_{k-1})I_{k-1}\cdot(\log\langle T\rangle)^{1/2}I_k$$
$$+ (\log\langle T\rangle VIII_{k-1} + VII_{k-1})(\log\langle T\rangle)^{1/2}I_{k-1}\cdot(\log\langle T\rangle)^{1/2}I_k. \tag{4.10}$$

Similarly for the second term from (4.7), we may use (3.16) and (3.17) to show

$$\int\int_{C_\tau^R} |\partial\Gamma^{\le 30}u_{k-1}||\Gamma^{\le 60}u_{k-1}|\left(|\partial\Gamma^{\le 60}u_k| + \frac{|\Gamma^{\le 60}u_k|}{r}\right)dxdt$$
$$\lesssim \left(\|\partial\Gamma^{\le 33}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + R\|\not\partial\partial\Gamma^{\le 32}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\right)\|r^{-1}\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^R)}$$
$$\times\left(\|\langle r\rangle^{-1}\partial\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^R)} + \|\langle r\rangle^{-1}r^{-1}\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^R)}\right)$$

and

$$\int\int_{C_\tau^U} |\partial\Gamma^{\le 30}u_{k-1}||\Gamma^{\le 60}u_{k-1}|\left(|\partial\Gamma^{\le 60}u_k| + \frac{|\Gamma^{\le 60}u_k|}{r}\right)dxdt$$
$$\lesssim U^{-1/2}\left(\tau^{-1/2}U^{1/2}\|\partial\Gamma^{\le 33}u_{k-1}\|_{L^2L^2(\tilde C_\tau^U)} + \tau^{1/2}U^{1/2}\|\bar\partial\partial\Gamma^{\le 32}u_{k-1}\|_{L^2L^2(\tilde C_\tau^U)}\right)$$
$$\times\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}$$
$$\times\left(\|\langle r\rangle^{-1/2}\partial\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^U)} + \|\langle r\rangle^{-1/2}r^{-1}\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^U)}\right).$$

Summing over R, U, and τ results in

$$\int_0^T\int |\partial\Gamma^{\le 30}u_{k-1}||\Gamma^{\le 60}u_{k-1}|\left(|\partial\Gamma^{\le 60}u_k| + \frac{|\Gamma^{\le 60}u_k|}{r}\right)dxdt \lesssim (IX_{k-1}+X_{k-1})VI_{k-1}I_k$$
$$+ (\log\langle T\rangle XI_{k-1} + \log\langle T\rangle XII_{k-1})\log\langle T\rangle VIII_{k-1}\cdot(\log\langle T\rangle)^{1/2}I_k. \quad (4.11)$$

For the third term from the right of (4.7), we bound

$$\int\int_{C_\tau^R} |\partial\Gamma^{\le 30}u_{k-1}||\partial\Gamma^{\le 60}u_{k-1}|\left(|\partial\Gamma^{\le 60}u_k| + \frac{|\Gamma^{\le 60}u_k|}{r}\right)dxdt$$
$$\lesssim \left(\|\partial\Gamma^{\le 33}u_{k-1}\|_{L^2L^2(\tilde C_\tau^R)} + R\|\bar\partial\partial\Gamma^{\le 32}u_{k-1}\|_{L^2L^2(\tilde C_\tau^R)}\right)\|\langle r\rangle^{-1}\partial\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^R)}$$
$$\times\left(\|\langle r\rangle^{-1}\partial\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^R)} + \|\langle r\rangle^{-1}r^{-1}\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^R)}\right)$$

using (3.16), and from (3.17), we have

$$\int\int_{C_\tau^U} |\partial\Gamma^{\le 30}u_{k-1}||\partial\Gamma^{\le 60}u_{k-1}|\left(|\partial\Gamma^{\le 60}u_k| + \frac{|\Gamma^{\le 60}u_k|}{r}\right)dxdt$$
$$\lesssim U^{-1}\left(\tau^{-1/2}U^{1/2}\|\partial\Gamma^{\le 33}u_{k-1}\|_{L^2L^2(\tilde C_\tau^U)} + \tau^{1/2}U^{1/2}\|\bar\partial\partial\Gamma^{\le 32}u_{k-1}\|_{L^2L^2(\tilde C_\tau^U)}\right)$$
$$\times\|\langle r\rangle^{-1/2}\partial\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}\left(\|\langle r\rangle^{-1/2}\partial\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^U)} + \|\langle r\rangle^{-1/2}r^{-1}\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^U)}\right).$$

Thus, by (1.20), (4.8), (4.9), and summing over $R\le\tau/4$, $U\le\tau/4$ and $\tau\le T$, it follows that

$$\int_0^T\int |\partial\Gamma^{\le 30}u_{k-1}||\partial\Gamma^{\le 60}u_{k-1}|\left(|\partial\Gamma^{\le 60}u_k| + \frac{|\Gamma^{\le 60}u_k|}{r}\right)dxdt \lesssim (IX_{k-1}+X_{k-1})I_{k-1}I_k$$
$$+ (\log\langle T\rangle XI_{k-1} + \log\langle T\rangle XII_{k-1})(\log\langle T\rangle)^{1/2}I_{k-1}\cdot(\log\langle T\rangle)^{1/2}I_k. \quad (4.12)$$

Together (4.6), (4.7), (4.10), (4.11), and (4.12) establish the desired bound (4.5).

4.1.2 Terms III_k through $VIII_k$

To control the next six terms of (4.2), we apply the local energy estimate (2.5) to $\Gamma^{\le 60}u_k$. Therefore,

$$III_k^2 + IV_k^2 + V_k^2 + VI_k^2 + VII_k^2 + VIII_k^2 \lesssim \|\partial\!\!\!/\Gamma^{\le 60}u_k(0,\cdot)\|_{L^2}^2 + \|r^{-1/2}\Gamma^{\le 60}u_k(0,\cdot)\|_{L^2}^2$$
$$+\|\Gamma^{\le 60}u_k\|_{LE^1}^2 + \|\partial\Gamma^{\le 60}u_k\|_{L^\infty L^2}^2 + \sup_{U\ge 1}\sup_{t\in[0,T]}\left|\int_0^t\int e^{-\sigma_U(t-r)}\Box\Gamma^{\le 60}u_k\cdot(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)dxdt\right|. \tag{4.13}$$

Using (1.5), (1.2), and a Hardy inequality, we can bound the first two terms on the right side of (4.13) by $(C_2\varepsilon)^2$ for a fixed constant C_2. Equation (4.5) provides the bound

$$\|\Gamma^{\le 60}u_k\|_{LE^1}^2 + \|\partial\Gamma^{\le 60}u_k\|_{L^\infty L^2}^2 \le (C_2\varepsilon)^2 + C(\log\langle T\rangle)^{5/2}M_{k-1}^2M_k. \tag{4.14}$$

In the work to follow, we will show

$$\sup_{U\ge 1}\sup_{t\in[0,T]}\left|\int_0^t\int e^{-\sigma_U(t-r)}\Box\Gamma^{\le 60}u_k\cdot(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)dxdt\right|$$
$$\lesssim (\log\langle T\rangle)^3M_{k-1}^2M_k + (\log\langle T\rangle)^5M_{k-1}^4. \tag{4.15}$$

Together (4.13), (4.14), and (4.15) establish the desired result for terms III_k through $VIII_k$.

In order to establish (4.15), using (1.10), we note that the most difficult term to bound will occur when all 60 vector fields land on a factor with a "bad" derivative $(\partial_t-\partial_r)$ when paired with a lower-order factor of u_{k-1}. We will deal with this term last. Thus, we set $\omega=(1,-x/r)$ and consider

$$|\Box\Gamma^{\le 60}u_k^I - a_{JK}^{I,\alpha}\omega_\alpha u_{k-1}^J(\partial_t-\partial_r)\Gamma^{\le 60}u_{k-1}^K| \lesssim |\Gamma^{\le 30}u_{k-1}||\partial\!\!\!/\Gamma^{\le 60}u_{k-1}| + |\Gamma^{\le 30}u_{k-1}||\partial\Gamma^{\le 59}u_{k-1}|$$
$$+|\Gamma^{\le 60}u_{k-1}||\partial\Gamma^{\le 30}u_{k-1}| + |\partial\Gamma^{\le 30}u_{k-1}||\partial\Gamma^{\le 60}u_{k-1}|. \tag{4.16}$$

Writing the integral as in (1.20), we will bound the integrals for each term in (4.16) over C_τ^R and C_τ^U and then sum over dyadic values of $R,U\le\tau/4$ and $\tau\le T$.

For the first term in the right side of (4.16), we apply (3.16) and (3.17) on C_τ^R and C_τ^U, respectively, to the factor with the least number of vector fields and the Schwarz inequality to obtain

$$\begin{aligned}\int\int_{C_\tau^R} &|\Gamma^{\le 30}u_{k-1}||\bar\partial\Gamma^{\le 60}u_{k-1}|(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt\\ &\lesssim \left(\tau^{-1/2}R^{-1/2}\|\Gamma^{\le 33}u_{k-1}\|_{L^2L^2(\tilde C_\tau^R)}+\tau^{-1/2}R^{1/2}\|(\partial_t+\partial_r)\Gamma^{\le 32}u_{k-1}\|_{L^2L^2(\tilde C_\tau^R)}\right)\\ &\qquad\times\|\bar\partial\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^R)}\left(\|\bar\partial\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^R)}+\|r^{-1}\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^R)}\right),\end{aligned}$$

where we used $r\lesssim R$ on $\tilde C_\tau^R$, and

$$\begin{aligned}\int\int_{C_\tau^U} &|\Gamma^{\le 30}u_{k-1}||\bar\partial\Gamma^{\le 60}u_{k-1}|(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt\\ &\lesssim \left(\tau^{-1/2}U^{-1/2}\|\Gamma^{33}u_{k-1}\|_{L^2L^2(\tilde C_\tau^U)}+\tau^{-1/2}U^{-1/2}\|(\partial_t+\partial_r)(r\Gamma^{\le 32}u_{k-1})\|_{L^2L^2(\tilde C_\tau^U)}\right)\\ &\qquad\times\|\bar\partial\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}\left(\|\bar\partial\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^U)}+\|r^{-1}\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^U)}\right).\end{aligned}$$

Additionally, on $\tilde C_\tau^R$, $\tau^{-1/2}\lesssim R^{-1/2}$ since $R\le\tau/4$ so that

$$\begin{aligned}\int\int_{C_\tau^R} &|\Gamma^{\le 30}u_{k-1}||\bar\partial\Gamma^{\le 60}u_{k-1}|(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt\\ &\lesssim \left(\|r^{-1}\Gamma^{33}u_{k-1}\|_{L^2L^2(\tilde C_\tau^R)}+\|(\partial_t+\partial_r)\Gamma^{\le 32}u_{k-1}\|_{L^2L^2(\tilde C_\tau^R)}\right)\|\bar\partial\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^R)}\\ &\qquad\times\left(\|\bar\partial\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^R)}+\|r^{-1}\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^R)}\right).\end{aligned}$$

On $\tilde C_\tau^U$, $\tau\approx r$ and $U\approx\langle t-r\rangle$ which gives

$$\begin{aligned}\int\int_{C_\tau^U} &|\Gamma^{\le 30}u_{k-1}||\bar\partial\Gamma^{\le 60}u_{k-1}|(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt\\ &\lesssim \left(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{33}u_{k-1}\|_{L^2L^2(\tilde C_\tau^U)}+\|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\le 32}u_{k-1})\|_{L^2L^2(\tilde C_\tau^U)}\right)\\ &\qquad\times\|\bar\partial\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}\left(\|\bar\partial\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^U)}+\|r^{-1}\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^U)}\right).\end{aligned}$$

After summing over R, U, and τ, we see

$$\int_0^T \int |\Gamma^{\leq 30} u_{k-1}||\bar{\partial}\Gamma^{\leq 60} u_{k-1}|(\partial_t + \partial_r)(r\Gamma^{\leq 60} u_k)|dxdt$$
$$\lesssim (VI_{k-1} + V_{k-1})V_{k-1}(V_k + VI_k) + (\log\langle T\rangle VIII_{k-1} + VII_{k-1})V_{k-1}(V_k + VI_k). \quad (4.17)$$

To handle the second term on the right of (4.16), we apply (3.3) to obtain

$$\int\int_{C_\tau^R} |\Gamma^{\leq 30} u_{k-1}||\partial\Gamma^{\leq 59} u_{k-1}||(\partial_t + \partial_r)(r\Gamma^{\leq 60} u_k)|dxdt$$
$$\lesssim \int\int_{C_\tau^R} |\Gamma^{\leq 30} u_{k-1}||\bar{\partial}\Gamma^{\leq 59} u_{k-1}||(\partial_t + \partial_r)(r\Gamma^{\leq 60} u_k)|dxdt$$
$$+ \int\int_{C_\tau^R} |\Gamma^{\leq 30} u_{k-1}|R^{-1}|\Gamma^{\leq 60} u_{k-1}||(\partial_t + \partial_r)(r\Gamma^{\leq 60} u_k)|dxdt. \quad (4.18)$$

The first integral on the right of (4.18) may be bounded using (4.17). For the second term, we apply (3.16) so that

$$\int\int_{C_\tau^R} |\Gamma^{\leq 30} u_{k-1}|R^{-1}|\Gamma^{\leq 60} u_{k-1}||(\partial_t + \partial_r)(r\Gamma^{\leq 60} u_k)|dxdt$$
$$\lesssim \left(\tau^{-1/2}R^{-1/2}\|\Gamma^{\leq 33} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + \tau^{-1/2}R^{1/2}\|(\partial_t + \partial_r)\Gamma^{\leq 32} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\right)$$
$$\times R^{-1}\|\Gamma^{\leq 60} u_{k-1}\|_{L^2L^2(C_\tau^R)}\left(\|\bar{\partial}\Gamma^{\leq 60} u_k\|_{L^2L^2(C_\tau^R)} + \|r^{-1}\Gamma^{\leq 60} u_k\|_{L^2L^2(C_\tau^R)}\right).$$

Therefore,

$$\int\int_{C_\tau^R} |\Gamma^{\leq 30} u_{k-1}|R^{-1}|\Gamma^{\leq 60} u_{k-1}||(\partial_t + \partial_r)(r\Gamma^{\leq 60} u_k)|dxdt$$
$$\lesssim \left(\|r^{-1}\Gamma^{\leq 33} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + \|\bar{\partial}\Gamma^{\leq 32} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\right)$$
$$\times \|r^{-1}\Gamma^{\leq 60} u_{k-1}\|_{L^2L^2(C_\tau^R)}\left(\|\bar{\partial}\Gamma^{\leq 60} u_k\|_{L^2L^2(C_\tau^R)} + \|r^{-1}\Gamma^{\leq 60} u_k\|_{L^2L^2(C_\tau^R)}\right). \quad (4.19)$$

Now we must bound the second term on the right of (4.16) when integrating over C_τ^U. Using

(3.4), we see

$$\begin{aligned}
\int\int_{C^U_\tau} &|\Gamma^{\le 30}u_{k-1}||\partial\Gamma^{\le 59}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt \\
&\lesssim \int\int_{C^U_\tau} |\Gamma^{\le 30}u_{k-1}||\bar\partial\Gamma^{\le 59}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt \\
&+ \int\int_{C^U_\tau} |\Gamma^{\le 30}u_{k-1}|U^{-1}|\Gamma^{\le 60}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt \\
&+ \int\int_{C^U_\tau} |\Gamma^{\le 30}u_{k-1}|U^{-1}|(\partial_t+\partial_r)(r\Gamma^{\le 59}u_{k-1})||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt. \quad (4.20)
\end{aligned}$$

The first term in the right of (4.20) can be controlled using (4.17). For the second and third terms of (4.20), applying (3.17) gives

$$\begin{aligned}
\int\int_{C^U_\tau} &|\Gamma^{\le 30}u_{k-1}|U^{-1}\left(|\Gamma^{\le 60}u_{k-1}|+|(\partial_t+\partial_r)(r\Gamma^{\le 59}u_{k-1})|\right)|(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt \\
&\lesssim \left(\tau^{-3/2}U^{-1/2}\|\Gamma^{\le 33}u_{k-1}\|_{L^2L^2(\tilde C^U_\tau)}+\tau^{-3/2}U^{-1/2}\|(\partial_t+\partial_r)(r\Gamma^{\le 32}u_{k-1})\|_{L^2L^2(\tilde C^U_\tau)}\right) \\
\times U^{-1}&\left(\|\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C^U_\tau)}+\|(\partial_t+\partial_r)(r\Gamma^{\le 59}u_{k-1})\|_{L^2L^2(C^U_\tau)}\right)\|(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)\|_{L^2L^2(C^U_\tau)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int\int_{C^U_\tau} &|\Gamma^{\le 30}u_{k-1}|U^{-1}\left(|\Gamma^{\le 60}u_{k-1}|+|(\partial_t+\partial_r)(r\Gamma^{\le 59}u_{k-1})|\right)|(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt \\
\lesssim &\left(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\le 33}u_{k-1}\|_{L^2L^2(\tilde C^U_\tau)}+\|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\le 32}u_{k-1})\|_{L^2L^2(\tilde C^U_\tau)}\right) \\
\times &\left(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C^U_\tau)}+\|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\le 59}u_{k-1})\|_{L^2L^2(C^U_\tau)}\right) \\
&\times\|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)\|_{L^2L^2(C^U_\tau)}. \quad (4.21)
\end{aligned}$$

Using (4.17), (4.18), (4.19), (4.20) and (4.21), and summing over $R, U\le \tau/4$ and $\tau\le T$, we conclude

$$\begin{aligned}
\int_0^T\int &|\Gamma^{\le 30}u_{k-1}||\partial\Gamma^{\le 59}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt \\
&\lesssim (V_{k-1}+VI_{k-1})^2(V_k+VI_k)+(\log\langle T\rangle VIII_{k-1}+VII_{k-1})V_{k-1}(V_k+VI_k) \\
&\quad+\log\langle T\rangle(\log\langle T\rangle VIII_{k-1}+VII_{k-1})^2VII_k. \quad (4.22)
\end{aligned}$$

As terms V_k and VI_k involve taking a supremum over U, the additional factor of $\log\langle T\rangle$ in the third term on the right side of (4.22) follows from summing over U.

Similarly, for the third term on the right in (4.16), we apply (3.16) and (3.17), the Schwarz inequality, and (1.14). This yields

$$\int\int_{C_\tau^R} |\Gamma^{\le 60}u_{k-1}||\partial\Gamma^{\le 30}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt \lesssim \|\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^R)}$$
$$\times\Big(\tau^{-1/2}R^{-1/2}\|\partial\Gamma^{\le 33}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}+\tau^{-1/2}R^{1/2}\|(\partial_t+\partial_r)\partial\Gamma^{\le 32}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big)$$
$$\times\Big(\|\partial\!\!\!/\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^R)}+\|r^{-1}\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^R)}\Big)$$

and respectively,

$$\int\int_{C_\tau^U} |\Gamma^{\le 60}u_{k-1}||\partial\Gamma^{\le 30}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt \lesssim \|\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}$$
$$\times\Big(\tau^{-3/2}U^{-1/2}\|\partial\Gamma^{\le 33}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}+\tau^{-3/2}U^{-1/2}\|(\partial_t+\partial_r)(r\partial\Gamma^{\le 32}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)$$
$$\times\|(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)\|_{L^2L^2(C_\tau^U)}.$$

As before, it follows that

$$\int\int_{C_\tau^R} |\Gamma^{\le 60}u_{k-1}||\partial\Gamma^{\le 30}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt \lesssim \|r^{-1}\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^R)}$$
$$\times\Big(\|\partial\Gamma^{\le 33}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}+R\|\partial\!\!\!/\partial\Gamma^{\le 32}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big)$$
$$\times\Big(\|\partial\!\!\!/\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^R)}+\|r^{-1}\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^R)}\Big)$$

and

$$\int\int_{C_\tau^U} |\Gamma^{\le 60}u_{k-1}||\partial\Gamma^{\le 30}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt \lesssim \|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}$$
$$\times\Big(\tau^{-1/2}U^{1/2}\|\partial\Gamma^{\le 33}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}+\tau^{1/2}U^{1/2}\|\partial\!\!\!/\partial\Gamma^{\le 32}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)$$
$$\times\|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)\|_{L^2L^2(C_\tau^U)}.$$

After summing, we see

$$\int_0^T \int |\Gamma^{\le 60} u_{k-1}||\partial \Gamma^{\le 30} u_{k-1}|(\partial_t + \partial_r)(r\Gamma^{\le 60} u_k)|dxdt$$
$$\lesssim VI_{k-1}\left(IX_{k-1} + X_{k-1}\right)\left(V_k + VI_k\right)$$
$$+ \log\langle T\rangle(\log\langle T\rangle VIII_{k-1}) \cdot \log\langle T\rangle(XI_{k-1} + XII_{k-1})VII_k. \quad (4.23)$$

Next we deal with the fourth term from (4.16). An application of (3.16) and (1.14) gives

$$\int\int_{C_\tau^R} |\partial\Gamma^{\le 30} u_{k-1}||\partial\Gamma^{\le 60} u_{k-1}||(\partial_t + \partial_r)(r\Gamma^{\le 60} u_k)|dxdt \lesssim \|\partial\Gamma^{\le 60} u_{k-1}\|_{L^2L^2(C_\tau^R)}$$
$$\times \left(\tau^{-1/2} R^{-1/2} \|\partial\Gamma^{\le 33} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + \tau^{-1/2} R^{1/2} \|(\partial_t + \partial_r)\partial\Gamma^{\le 32} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\right)$$
$$\times \left(\|\not\partial\Gamma^{\le 60} u_k\|_{L^2L^2(C_\tau^R)} + \|r^{-1}\Gamma^{\le 60} u_k\|_{L^2L^2(C_\tau^R)}\right).$$

Using (3.17) instead results in

$$\int\int_{C_\tau^U} |\partial\Gamma^{\le 30} u_{k-1}||\partial\Gamma^{\le 60} u_{k-1}|(\partial_t + \partial_r)(r\Gamma^{\le 60} u_k)|dxdt \lesssim \|\partial\Gamma^{\le 60} u_{k-1}\|_{L^2L^2(C_\tau^U)}$$
$$\times \left(\tau^{-3/2} U^{-1/2} \|\partial\Gamma^{\le 33} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \tau^{-3/2} U^{-1/2} \|(\partial_t + \partial_r)(r\partial\Gamma^{\le 32} u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\right)$$
$$\times \|(\partial_t + \partial_r)(r\Gamma^{\le 60} u_k)\|_{L^2L^2(C_\tau^U)}.$$

As a direct consequence, we have

$$\int\int_{C_\tau^R} |\partial\Gamma^{\le 30} u_{k-1}||\partial\Gamma^{\le 60} u_{k-1}||(\partial_t + \partial_r)(r\Gamma^{\le 60} u_k)|dxdt \lesssim R^{-1}\|\partial\Gamma^{\le 60} u_{k-1}\|_{L^2L^2(C_\tau^R)}$$
$$\times \left(\|\partial\Gamma^{\le 33} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + R\|\not\partial\partial\Gamma^{\le 32} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\right)$$
$$\times \left(\|\not\partial\Gamma^{\le 60} u_k\|_{L^2L^2(C_\tau^R)} + \|r^{-1}\Gamma^{\le 60} u_k\|_{L^2L^2(C_\tau^R)}\right). \quad (4.24)$$

Additionally,

$$\int\int_{C_\tau^U} |\partial\Gamma^{\le 30}u_{k-1}||\partial\Gamma^{\le 60}u_{k-1}|(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt \lesssim \tau^{-1/2}U^{-1/2}\|\partial\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}$$
$$\times\Big(\tau^{-1/2}U^{1/2}\|\partial\Gamma^{\le 33}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}+\tau^{1/2}U^{1/2}\|\bar{\partial}\partial\Gamma^{\le 32}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)$$
$$\times\,\tau^{-1/2}U^{-1/2}\|(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)\|_{L^2L^2(C_\tau^U)}. \quad (4.25)$$

Therefore, summing (4.24) and (4.25) using (4.8) and (4.9), respectively, we find

$$\int_0^T\int |\partial\Gamma^{\le 30}u_{k-1}||\partial\Gamma^{\le 60}u_{k-1}|(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt$$
$$\lesssim I_{k-1}(IX_{k-1}+X_{k-1})(V_k+VI_k)$$
$$+(\log\langle T\rangle)^{1/2}I_{k-1}\cdot\log\langle T\rangle(XI_{k-1}+XII_{k-1})VII_k. \quad (4.26)$$

From here, we will show

$$\sup_{U\ge 1}\sup_{t\in[0,T]}\left|\int_0^t\int e^{-\sigma_U(t-r)}a^{I,\alpha}_{JK}\omega_\alpha u^J_{k-1}(\partial_t-\partial_r)\Gamma^{\le 60}u^K_{k-1}(\partial_t+\partial_r)(r\Gamma^{\le 60}u^I_k)dxdt\right|$$
$$\lesssim(\log\langle T\rangle)^3M_{k-1}^2M_k+(\log\langle T\rangle)^5M_{k-1}^4. \quad (4.27)$$

Together with (4.16), (4.17), (4.22), (4.23), and (4.26), this finishes the proof of (4.15) which completes the desired bound for terms III_k through $VIII_k$.

Towards establishing (4.27), we convert to spherical coordinates and integrate by parts in $\partial_t-\partial_r$

so that

$$\begin{aligned}
\int_0^t \int e^{-\sigma_U(t-r)} a^{I,\alpha}_{JK}\omega_\alpha u^J_{k-1}(\partial_t-\partial_r)\Gamma^{\le 60}u^K_{k-1}(\partial_t+\partial_r)(r\Gamma^{\le 60}u^I_k)\,dxdt \\
= \int_{\mathbb{R}^3} e^{-\sigma_U(t-r)} a^{I,\alpha}_{JK}\omega_\alpha u^J_{k-1}\Gamma^{\le 60}u^K_{k-1}(\partial_t+\partial_r)(r\Gamma^{\le 60}u^I_k)\,dx\Big|_0^t \\
+2\int_0^t\int r^{-1}e^{-\sigma_U(t-r)} a^{I,\alpha}_{JK}\omega_\alpha u^J_{k-1}\Gamma^{\le 60}u^K_{k-1}(\partial_t+\partial_r)(r\Gamma^{\le 60}u^I_k)\,dxdt \\
+2\int_0^t\int e^{-\sigma_U(t-r)}\sigma_U'(t-r) a^{I,\alpha}_{JK}\omega_\alpha u^J_{k-1}\Gamma^{\le 60}u^K_{k-1}(\partial_t+\partial_r)(r\Gamma^{\le 60}u^I_k)\,dxdt \\
-\int_0^t\int e^{-\sigma_U(t-r)} a^{I,\alpha}_{JK}\omega_\alpha(\partial_t-\partial_r)u^J_{k-1}\Gamma^{\le 60}u^K_{k-1}(\partial_t+\partial_r)(r\Gamma^{\le 60}u^I_k)\,dxdt \\
-\int_0^t\int e^{-\sigma_U(t-r)} a^{I,\alpha}_{JK}\omega_\alpha u^J_{k-1}\Gamma^{\le 60}u^K_{k-1}(\partial_t^2-\partial_r^2)(r\Gamma^{\le 60}u^I_k)\,dxdt. \quad (4.28)
\end{aligned}$$

To bound the first term in the right side of (4.28), we fix $t\in[0,T]$ and take $\tau\approx t$. Then,

$$\begin{aligned}
\int |u_{k-1}||\Gamma^{\le 60}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|\,dx \lesssim \sum_{R\le\tau/4}\int_{A_R}|u_{k-1}||\Gamma^{\le 60}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|\,dx \\
+\sum_{U\le\tau/4}\int_{\langle t-r\rangle\approx U}|u_{k-1}||\Gamma^{\le 60}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|\,dx. \quad (4.29)
\end{aligned}$$

Using (3.16), we find

$$\begin{aligned}
\int_{A_R}|u_{k-1}||\Gamma^{\le 60}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|\,dx \\
\lesssim \Big(\tau^{-1/2}R^{-3/2}\|\Gamma^{\le 3}u_{k-1}\|_{L^2L^2(\tilde C^R_\tau)}+\tau^{-1/2}R^{-1/2}\|(\partial_t+\partial_r)\Gamma^{\le 2}u_{k-1}\|_{L^2L^2(\tilde C^R_\tau)}\Big) \\
\times\int_{A_R} r|\Gamma^{\le 60}u_{k-1}|\left(|\bar\partial\Gamma^{\le 60}u_k|+r^{-1}|\Gamma^{\le 60}u_k|\right)dx.
\end{aligned}$$

Applying the Schwarz inequality and summing over $R\le\tau/4$ yields

$$\begin{aligned}
\sum_{R\le\tau/4}\int_{A_R}|u_{k-1}||\Gamma^{\le 60}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|\,dx \\
\lesssim \sum_{R\le\tau/4}\Big(\|r^{-1}\Gamma^{\le 3}u_{k-1}\|_{L^2L^2(\tilde C^R_\tau)}+\|\bar\partial\Gamma^{\le 2}u_{k-1}\|_{L^2L^2(\tilde C^R_\tau)}\Big)\|r^{-1/2}\Gamma^{\le 60}u_{k-1}(t,\cdot)\|_{L^2(A_R)} \\
\times\Big(\|\langle r\rangle^{1/2}\bar\partial\Gamma^{\le 60}u_k(t,\cdot)\|_{L^2(A_R)}+\|r^{-1/2}\Gamma^{\le 60}u_k(t,\cdot)\|_{L^2(A_R)}\Big) \quad (4.30) \\
\lesssim (VI_{k-1}+V_{k-1})IV_{k-1}(III_k+IV_k).
\end{aligned}$$

Next, by applying (3.17), we have

$$\int_{\langle t-r\rangle\approx U} |u_{k-1}||\Gamma^{\le 60}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dx$$
$$\lesssim \Big(\tau^{-3/2}U^{-1/2}\|\Gamma^{\le 3}u_{k-1}\|_{L^2L^2(\tilde{C}^U_\tau)} + \tau^{-3/2}U^{-1/2}\|(\partial_t+\partial_r)(r\Gamma^{\le 2}u_{k-1})\|_{L^2L^2(\tilde{C}^U_\tau)}\Big)$$
$$\times \int_{\langle t-r\rangle\approx U} r|\Gamma^{\le 60}u_{k-1}|\left(|\bar{\partial}\Gamma^{\le 60}u_k| + r^{-1}|\Gamma^{\le 60}u_k|\right)dx.$$

Therefore,

$$\sum_{U\le\tau/4}\int_{\langle t-r\rangle\approx U} |u_{k-1}||\Gamma^{\le 60}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dx$$
$$\lesssim \sum_{U\le\tau/4}\Big(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\le 3}u_{k-1}\|_{L^2L^2(\tilde{C}^U_\tau)} + \|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\le 2}u_{k-1})\|_{L^2L^2(\tilde{C}^U_\tau)}\Big)$$
$$\times \|r^{-1/2}\Gamma^{\le 60}u_{k-1}(t,\cdot)\|_{L^2(\langle t-r\rangle\approx U)}$$
$$\times \Big(\|\langle r\rangle^{1/2}\bar{\partial}\Gamma^{\le 60}u_k(t,\cdot)\|_{L^2(\langle t-r\rangle\approx U)} + \|r^{-1/2}\Gamma^{\le 60}u_k(t,\cdot)\|_{L^2(\langle t-r\rangle\approx U)}\Big)$$
$$\lesssim \sup_{U\ge 1}\Big(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\le 3}u_{k-1}\|_{L^2L^2(\tilde{C}^U_\tau)} + \|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\le 2}u_{k-1})\|_{L^2L^2(\tilde{C}^U_\tau)}\Big)$$
$$\times \|r^{-1/2}\Gamma^{\le 60}u_{k-1}(t,\cdot)\|_{L^2}\Big(\|\langle r\rangle^{1/2}\bar{\partial}\Gamma^{\le 60}u_k(t,\cdot)\|_{L^2} + \|r^{-1/2}\Gamma^{\le 60}u_k(t,\cdot)\|_{L^2}\Big). \quad (4.31)$$

Together (4.29), (4.30), and (4.31) show

$$\sup_{U\ge 1}\sup_{t\in[0,T]}\left|\int e^{-\sigma_U(t-r)}a^{I,\alpha}_{JK}\omega_\alpha u^J_{k-1}\Gamma^{\le 60}u^K_{k-1}(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)dx\Big|_0^t\right|$$
$$\lesssim (VI_{k-1}+V_{k-1}+\log\langle T\rangle VIII_{k-1}+VII_{k-1})IV_{k-1}(III_k+IV_k). \quad (4.32)$$

For the second term in the right side of (4.28), we may bound

$$\int_0^t\int r^{-1}|u_{k-1}||\Gamma^{\le 60}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt$$
$$\lesssim \sum_{\tau\le t}\Big(\sum_{R\le\tau/4}\int\int_{C^R_\tau} r^{-1}|u_{k-1}||\Gamma^{\le 60}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt$$
$$+\sum_{U\le\tau/4}\int\int_{C^U_\tau} r^{-1}|u_{k-1}||\Gamma^{\le 60}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt\Big)$$

as in (1.20). To control the integrals over C_τ^R and C_τ^U, we apply (3.16) and (3.17) to u_{k-1} and the Schwarz inequality so that

$$\begin{aligned}
&\int\int_{C_\tau^R} r^{-1}|u_{k-1}||\Gamma^{\le 60}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt\\
&\quad\lesssim \left(\tau^{-1/2}R^{-3/2}\|\Gamma^{\le 3}u_{k-1}\|_{L^2L^2(\tilde C_\tau^R)}+\tau^{-1/2}R^{-1/2}\|(\partial_t+\partial_r)\Gamma^{\le 2}u_{k-1}\|_{L^2L^2(\tilde C_\tau^R)}\right)\\
&\qquad\times\|r^{-1}\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^R)}\left(R\|(\partial_t+\partial_r)\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^R)}+\|\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^R)}\right)
\end{aligned}$$

and respectively,

$$\begin{aligned}
&\int\int_{C_\tau^U} r^{-1}|u_{k-1}||\Gamma^{\le 60}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt\\
&\quad\lesssim \left(\tau^{-3/2}U^{-1/2}\|\Gamma^{\le 3}u_{k-1}\|_{L^2L^2(\tilde C_\tau^U)}+\tau^{-3/2}U^{-1/2}\|(\partial_t+\partial_r)(r\Gamma^{\le 2}u_{k-1}\|_{L^2L^2(\tilde C_\tau^U)}\right)\\
&\qquad\times\|r^{-1}\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}\|(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)\|_{L^2L^2(C_\tau^U)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int\int_{C_\tau^R} r^{-1}|u_{k-1}||\Gamma^{\le 60}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt\\
&\quad\lesssim \left(\|r^{-1}\Gamma^{\le 3}u_{k-1}\|_{L^2L^2(\tilde C_\tau^R)}+\|\bar\partial\Gamma^{\le 2}u_{k-1}\|_{L^2L^2(\tilde C_\tau^R)}\right)\|r^{-1}\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^R)}\\
&\qquad\times\left(\|\bar\partial\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^R)}+\|r^{-1}\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^R)}\right) \qquad (4.33)
\end{aligned}$$

and

$$\begin{aligned}
&\int\int_{C_\tau^U} r^{-1}|u_{k-1}||\Gamma^{\le 60}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\le 60}u_k)|dxdt\\
&\quad\lesssim \left(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\le 3}u_{k-1}\|_{L^2L^2(\tilde C_\tau^U)}+\|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\le 2}u_{k-1}\|_{L^2L^2(\tilde C_\tau^U)}\right)\\
&\qquad\times\|r^{-1}\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}\left(\|\bar\partial\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^U)}+\|r^{-1}\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^U)}\right).
\end{aligned}$$

By summing over $R, U \leq \tau/4$ and $\tau \leq t$, we see

$$\sup_{U\geq 1}\sup_{t\in[0,T]}\left|\int_0^t\int r^{-1}e^{-\sigma_U(t-r)}a^{I,\alpha}_{JK}\omega_\alpha u^J_{k-1}\Gamma^{\leq 60}u^K_{k-1}(\partial_t+\partial_r)(r\Gamma^{\leq 60}u^I_k)dxdt\right|$$
$$\lesssim (VI_{k-1}+V_{k-1}+\log\langle T\rangle VIII_{k-1}+VII_{k-1})VI_{k-1}(V_k+VI_k). \quad (4.34)$$

To handle the third term from (4.28), we will use a nearly identical argument. Since $\sigma'_U(t-r)\lesssim \tau^{-1}\lesssim R^{-1}$ on C^R_τ, we may directly use (4.33). For the integral over C^U_τ, we note $\sigma'_U(t-r)\lesssim U^{-1}$ and apply (3.17) so that

$$\begin{aligned}\int\int_{C^U_\tau}&|\sigma'_U(t-r)||u_{k-1}||\Gamma^{\leq 60}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\leq 60}u_k)|dxdt\\
&\lesssim U^{-1}\Big(\tau^{-3/2}U^{-1/2}\|\Gamma^{\leq 3}u_{k-1}\|_{L^2L^2(\tilde C^U_\tau)}+\tau^{-3/2}U^{-1/2}\|(\partial_t+\partial_r)(r\Gamma^{\leq 2}u_{k-1})\|_{L^2L^2(\tilde C^U_\tau)}\Big)\\
&\qquad\times\|\Gamma^{\leq 60}u_{k-1}\|_{L^2L^2(C^U_\tau)}\|(\partial_t+\partial_r)(r\Gamma^{\leq 60}u_k)\|_{L^2L^2(C^U_\tau)}.\end{aligned}$$

Then,

$$\begin{aligned}\int\int_{C^U_\tau}&|\sigma'_U(t-r)||u_{k-1}||\Gamma^{\leq 60}u_{k-1}||(\partial_t+\partial_r)(r\Gamma^{\leq 60}u_k)|dxdt\\
&\lesssim\Big(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\leq 3}u_{k-1}\|_{L^2L^2(\tilde C^U_\tau)}+\|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\leq 2}u_{k-1})\|_{L^2L^2(\tilde C^U_\tau)}\Big)\\
&\qquad\times\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\leq 60}u_{k-1}\|_{L^2L^2(C^U_\tau)}\|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\leq 60}u_k)\|_{L^2L^2(C^U_\tau)}.\end{aligned}$$

We conclude

$$\sup_{U\geq 1}\sup_{t\in[0,T]}\left|\int_0^t\int e^{-\sigma_U(t-r)}\sigma'_U(t-r)a^{I,\alpha}_{JK}\omega_\alpha u^J_{k-1}\Gamma^{\leq 60}u^K_{k-1}(\partial_t+\partial_r)(r\Gamma^{\leq 60}u^I_k)dxdt\right|$$
$$\lesssim (VI_{k-1}+V_{k-1})VI_{k-1}(V_k+VI_k)+\log\langle T\rangle(\log\langle T\rangle VIII_{k-1}+VII_{k-1})\log\langle T\rangle VIII_{k-1}VII_k \quad (4.35)$$

where the additional factor of $\log\langle T\rangle$ in the second term on the right of (4.35) comes from summing over U.

It suffices to use (4.23) to bound the fourth term on the right side of (4.28).

For the final term of (4.28), we use (2.11) so that

$$\int_0^t \int e^{-\sigma_U(t-r)} a^{I,\alpha}_{JK} \omega_\alpha u^J_{k-1} \Gamma^{\le 60} u^K_{k-1} (\partial_t^2 - \partial_r^2)(r \Gamma^{\le 60} u^I_k) dx dt$$
$$= \int_0^t \int r e^{-\sigma_U(t-r)} a^{I,\alpha}_{JK} \omega_\alpha u^J_{k-1} \Gamma^{\le 60} u^K_{k-1} \Box \Gamma^{\le 60} u^I_k dx dt$$
$$+ \int_0^t \int r e^{-\sigma_U(t-r)} a^{I,\alpha}_{JK} \omega_\alpha u^J_{k-1} \Gamma^{\le 60} u^K_{k-1} \not\nabla \cdot \not\nabla \Gamma^{\le 60} u^I_k dx dt. \quad (4.36)$$

We will now create a bound for both terms on the right side of (4.36) in order to show

$$\sup_{U\ge 1} \sup_{t\in[0,T]} \left| \int_0^t \int e^{-\sigma_U(t-r)} a^{I,\alpha}_{JK} \omega_\alpha u^J_{k-1} \Gamma^{\le 60} u^K_{k-1} (\partial_t^2 - \partial_r^2)(r \Gamma^{\le 60} u^I_k) dx dt \right|$$
$$\lesssim \log\langle T\rangle M_{k-1}^2 M_k + (\log\langle T\rangle)^5 M_{k-1}^4. \quad (4.37)$$

Together with (4.28), (4.32), (4.34), (4.35), and (4.23), this completes the proof of (4.27).

It remains to look at both terms on the right side of (4.36). We begin with the second term. Integration by parts in $\not\nabla$ yields

$$\int_0^t \int r e^{-\sigma_U(t-r)} a^{I,\alpha}_{JK} \omega_\alpha u^J_{k-1} \Gamma^{\le 60} u^K_{k-1} \not\nabla \cdot \not\nabla \Gamma^{\le 60} u^I_k dx dt$$
$$= \int_0^t \int e^{-\sigma_U(t-r)} a^{I,j}_{JK} u^J_{k-1} \Gamma^{\le 60} u^K_{k-1} \not\nabla_j \Gamma^{\le 60} u^I_k dx dt$$
$$- \int_0^t \int r e^{-\sigma_U(t-r)} a^{I,\alpha}_{JK} \omega_\alpha \Gamma^{\le 60} u^K_{k-1} \not\nabla u^J_{k-1} \cdot \not\nabla \Gamma^{\le 60} u^I_k dx dt$$
$$- \int_0^t \int r e^{-\sigma_U(t-r)} a^{I,\alpha}_{JK} \omega_\alpha u^J_{k-1} \not\nabla \Gamma^{\le 60} u^K_{k-1} \cdot \not\nabla \Gamma^{\le 60} u^I_k dx dt.$$

Using (1.9), we see

$$\left| \int_0^t \int r e^{-\sigma_U(t-r)} a^{I,\alpha}_{JK} \omega_\alpha u^J_{k-1} \Gamma^{\le 60} u^K_{k-1} \not\nabla \cdot \not\nabla \Gamma^{\le 60} u^I_k dx dt \right|$$
$$\lesssim \int_0^t \int |\Gamma^{\le 60} u_{k-1}| |\Gamma^{\le 1} u_{k-1}| |\not\nabla \Gamma^{\le 60} u_k| dx dt + \int_0^t \int r |u_{k-1}| |\not\nabla \Gamma^{\le 60} u_{k-1}| |\not\nabla \Gamma^{\le 60} u_k| dx dt.$$

For $R \leq \tau/4$ and $U \leq \tau/4$, applications of (3.16) and (3.17) yield

$$\begin{aligned}
&\int\int_{C_\tau^R} |\Gamma^{\leq 60} u_{k-1}||\Gamma^{\leq 1} u_{k-1}||\bar{\nabla}\Gamma^{\leq 60} u_k| dxdt + \int\int_{C_\tau^R} r|u_{k-1}||\bar{\nabla}\Gamma^{\leq 60} u_{k-1}||\bar{\nabla}\Gamma^{\leq 60} u_k| dxdt \\
&\quad\lesssim \Big(\tau^{-1/2} R^{-3/2} \|\Gamma^{\leq 4} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + \tau^{-1/2} R^{-1/2} \|(\partial_t + \partial_r)\Gamma^{\leq 3} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big) \\
&\qquad\qquad \times \|\Gamma^{\leq 60} u_{k-1}\|_{L^2L^2(C_\tau^R)} \|\bar{\nabla}\Gamma^{\leq 60} u_k\|_{L^2L^2(C_\tau^R)} \\
&\quad + \Big(\tau^{-1/2} R^{-1/2} \|\Gamma^{\leq 3} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + \tau^{-1/2} R^{1/2} \|(\partial_t + \partial_r)\Gamma^{\leq 2} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big) \\
&\qquad\qquad \times \|\bar{\nabla}\Gamma^{\leq 60} u_{k-1}\|_{L^2L^2(C_\tau^R)} \|\bar{\nabla}\Gamma^{\leq 60} u_k\|_{L^2L^2(C_\tau^R)}
\end{aligned}$$

and

$$\begin{aligned}
&\int\int_{C_\tau^U} |\Gamma^{\leq 60} u_{k-1}||\Gamma^{\leq 1} u_{k-1}||\bar{\nabla}\Gamma^{\leq 60} u_k| dxdt + \int\int_{C_\tau^U} r|u_{k-1}||\bar{\nabla}\Gamma^{\leq 60} u_{k-1}||\bar{\nabla}\Gamma^{\leq 60} u_k| dxdt \\
&\quad\lesssim \Big(\tau^{-3/2} U^{-1/2} \|\Gamma^{\leq 4} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \tau^{-3/2} U^{-1/2} \|(\partial_t + \partial_r)(r\Gamma^{\leq 3} u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\Big) \\
&\qquad\qquad \times \|\Gamma^{\leq 60} u_{k-1}\|_{L^2L^2(C_\tau^U)} \|\bar{\nabla}\Gamma^{\leq 60} u_k\|_{L^2L^2(C_\tau^U)} \\
&\quad + \Big(\tau^{-1/2} U^{-1/2} \|\Gamma^{\leq 3} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \tau^{-1/2} U^{-1/2} \|(\partial_t + \partial_r)(r\Gamma^{\leq 2} u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\Big) \\
&\qquad\qquad \times \|\bar{\nabla}\Gamma^{\leq 60} u_{k-1}\|_{L^2L^2(C_\tau^U)} \|\bar{\nabla}\Gamma^{\leq 60} u_k\|_{L^2L^2(C_\tau^U)},
\end{aligned}$$

respectively. Immediately, it follows that

$$\begin{aligned}
&\int\int_{C_\tau^R} |\Gamma^{\leq 60} u_{k-1}||\Gamma^{\leq 1} u_{k-1}||\bar{\nabla}\Gamma^{\leq 60} u_k| dxdt + \int\int_{C_\tau^R} r|u_{k-1}||\bar{\nabla}\Gamma^{\leq 60} u_{k-1}||\bar{\nabla}\Gamma^{\leq 60} u_k| dxdt \\
&\lesssim \Big(\|r^{-1}\Gamma^{\leq 4} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + \|\bar{\partial}\Gamma^{\leq 3} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big) \|r^{-1}\Gamma^{\leq 60} u_{k-1}\|_{L^2L^2(C_\tau^R)} \|\bar{\partial}\Gamma^{\leq 60} u_k\|_{L^2L^2(C_\tau^R)} \\
&\quad + \Big(\|r^{-1}\Gamma^{\leq 3} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + \|\bar{\partial}\Gamma^{\leq 2} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big) \|\bar{\partial}\Gamma^{\leq 60} u_{k-1}\|_{L^2L^2(C_\tau^R)} \|\bar{\partial}\Gamma^{\leq 60} u_k\|_{L^2L^2(C_\tau^R)}
\end{aligned}$$

and

$$\int\int_{C_\tau^U} |\Gamma^{\le 60}u_{k-1}||\Gamma^{\le 1}u_{k-1}||\bar\partial\Gamma^{\le 60}u_k|dxdt + \int\int_{C_\tau^U} r|u_{k-1}||\bar\partial\Gamma^{\le 60}u_{k-1}||\bar\partial\Gamma^{\le 60}u_k|dxdt$$
$$\lesssim \Big(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\le 4}u_{k-1}\|_{L^2L^2(\tilde C_\tau^U)} + \|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\le 3}u_{k-1})\|_{L^2L^2(\tilde C_\tau^U)}\Big)$$
$$\|r^{-1}\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}\|\bar\partial\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^U)}$$
$$+\Big(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\le 3}u_{k-1}\|_{L^2L^2(\tilde C_\tau^U)} + \|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\le 2}u_{k-1})\|_{L^2L^2(\tilde C_\tau^U)}\Big)$$
$$\|\bar\partial\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}\|\bar\partial\Gamma^{\le 60}u_k\|_{L^2L^2(C_\tau^U)}.$$

A summation over $R \le \tau/4$, $U \le \tau/4$ and $\tau \le t$ yields

$$\sup_{U\ge 1}\sup_{t\in[0,T]}\Big|\int_0^t\int re^{-\sigma_U(t-r)}a^{I,\alpha}_{JK}\omega_\alpha u^J_{k-1}\Gamma^{\le 60}u^K_{k-1}\bar\nabla\cdot\bar\nabla u^I_k dxdt\Big|$$
$$\lesssim (VI_{k-1}+V_{k-1})^2V_k + (\log\langle T\rangle VIII_{k-1} + VII_{k-1})(VI_{k-1}+V_{k-1})V_k, \quad (4.38)$$

which is the desired bound for the second term on the right side of (4.36).

To handle the first term on the right of (4.36), a more delicate argument will be required for terms of $\Box\Gamma^{\le 60}u_k$ of the form $u_{k-1}(\partial_t-\partial_r)\Gamma^{\le 60}u_{k-1}$. Therefore, using (4.16), we first consider

$$\sup_{U\ge 1}\sup_{t\in[0,T]}\Big|\int_0^t\int re^{-\sigma_U(t-r)}a^{I,\alpha}_{JK}\omega_\alpha u^J_{k-1}\Gamma^{\le 60}u^K_{k-1}$$
$$\times\Big(\Box\Gamma^{\le 60}u^I_k - a^{I,\beta}_{PQ}\omega_\beta u^P_{k-1}(\partial_t-\partial_r)\Gamma^{\le 60}u^Q_{k-1}\Big)dxdt\Big|$$
$$\lesssim \int_0^T\int r|u_{k-1}||\Gamma^{\le 60}u_{k-1}||\Gamma^{\le 30}u_{k-1}||\bar\partial\Gamma^{\le 60}u_{k-1}|dxdt$$
$$+\int_0^T\int r|u_{k-1}||\Gamma^{\le 60}u_{k-1}||\Gamma^{\le 30}u_{k-1}||\partial\Gamma^{\le 59}u_{k-1}|dxdt$$
$$+\int_0^T\int r|u_{k-1}||\Gamma^{\le 60}u_{k-1}|^2|\partial\Gamma^{\le 30}u_{k-1}|dxdt$$
$$+\int_0^T\int r|u_{k-1}||\Gamma^{\le 60}u_{k-1}||\partial\Gamma^{\le 30}u_{k-1}||\partial\Gamma^{\le 60}u_{k-1}|dxdt. \quad (4.39)$$

The terms above are once again readily controlled using (1.20), the Schwarz inequality, (3.16), and (3.17) as we did in establishing (4.17), (4.22), (4.23), and (4.26).

That is, for the first term on the right of (4.39), the Schwarz inequality and two applications of

(3.16) give

$$\begin{aligned}\int\int_{C^R_\tau} r|u_{k-1}||\Gamma^{\le 60}u_{k-1}||\Gamma^{\le 30}u_{k-1}||\bar\partial\Gamma^{\le 60}u_{k-1}|dxdt \\ \lesssim \Big(\|r^{-1}\Gamma^{\le 3}u_{k-1}\|_{L^2L^2(\tilde C^R_\tau)} + \|\bar\partial\Gamma^{\le 2}u_{k-1}\|_{L^2L^2(\tilde C^R_\tau)}\Big)\|r^{-1}\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C^R_\tau)} \\ \times\Big(\|r^{-1}\Gamma^{\le 33}u_{k-1}\|_{L^2L^2(\tilde C^R_\tau)} + \|\bar\partial\Gamma^{\le 32}u_{k-1}\|_{L^2L^2(\tilde C^R_\tau)}\Big)\|\bar\partial\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C^R_\tau)}.\end{aligned}$$

Similarly, using (3.17), we see

$$\begin{aligned}\int\int_{C^U_\tau} r|u_{k-1}||\Gamma^{\le 60}u_{k-1}||\Gamma^{\le 30}u_{k-1}||\bar\partial\Gamma^{\le 60}u_{k-1}|dxdt \\ \lesssim \Big(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\le 3}u_{k-1}\|_{L^2L^2(\tilde C^U_\tau)} + \|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\le 2}u_{k-1})\|_{L^2L^2(\tilde C^U_\tau)}\Big) \\ \times\Big(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\le 33}u_{k-1}\|_{L^2L^2(\tilde C^U_\tau)} + \|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\le 32}u_{k-1})\|_{L^2L^2(\tilde C^U_\tau)}\Big) \\ \times\|r^{-1}\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C^U_\tau)}\|\bar\partial\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C^U_\tau)}.\end{aligned}$$

It follows that

$$\begin{aligned}\int_0^T\int r|u_{k-1}||\Gamma^{\le 60}u_{k-1}||\Gamma^{\le 30}u_{k-1}||\bar\partial\Gamma^{\le 60}u_{k-1}|dxdt \\ \lesssim (VI_{k-1}+V_{k-1})^2 VI_{k-1}V_{k-1} + (\log\langle T\rangle VIII_{k-1} + VII_{k-1})^2 VI_{k-1}V_{k-1}.\end{aligned}\tag{4.40}$$

Using this same argument for the second term on the right of (4.39), we have

$$\begin{aligned}\int\int_{C^R_\tau} r|u_{k-1}||\Gamma^{\le 60}u_{k-1}||\Gamma^{\le 30}u_{k-1}||\partial\Gamma^{\le 59}u_{k-1}|dxdt \\ \lesssim \Big(\|r^{-1}\Gamma^{\le 3}u_{k-1}\|_{L^2L^2(\tilde C^R_\tau)} + \|\bar\partial\Gamma^{\le 2}u_{k-1}\|_{L^2L^2(\tilde C^R_\tau)}\Big)\|r^{-1}\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C^R_\tau)} \\ \times\Big(\|r^{-1}\Gamma^{\le 33}u_{k-1}\|_{L^2L^2(\tilde C^R_\tau)} + \|\bar\partial\Gamma^{\le 32}u_{k-1}\|_{L^2L^2(\tilde C^R_\tau)}\Big)\|\partial\Gamma^{\le 59}u_{k-1}\|_{L^2L^2(C^R_\tau)}\end{aligned}$$

and

$$
\begin{aligned}
&\int\int_{C_\tau^U} r|u_{k-1}||\Gamma^{\leq 60}u_{k-1}||\Gamma^{\leq 30}u_{k-1}||\partial\Gamma^{\leq 59}u_{k-1}|dxdt \\
&\quad\lesssim \left(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\leq 3}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\leq 2}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\right) \\
&\quad\times \left(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\leq 33}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\leq 32}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\right) \\
&\qquad\qquad\qquad\qquad\times \|r^{-1/2}\Gamma^{\leq 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}\|r^{-1/2}\partial\Gamma^{\leq 59}u_{k-1}\|_{L^2L^2(C_\tau^U)}.
\end{aligned}
$$

Then we apply (3.3) and (3.4) to the factor involving $|\partial\Gamma^{\leq 59}u_{k-1}|$ so that

$$
\begin{aligned}
&\int\int_{C_\tau^R} r|u_{k-1}||\Gamma^{\leq 60}u_{k-1}||\Gamma^{\leq 30}u_{k-1}||\partial\Gamma^{\leq 59}u_{k-1}|dxdt \\
&\qquad\lesssim \left(\|r^{-1}\Gamma^{\leq 3}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + \|\not\partial\Gamma^{\leq 2}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\right)\|r^{-1}\Gamma^{\leq 60}u_{k-1}\|_{L^2L^2(C_\tau^R)} \\
&\qquad\qquad\times \left(\|r^{-1}\Gamma^{\leq 33}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + \|\not\partial\Gamma^{\leq 32}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\right) \\
&\qquad\qquad\qquad\left(\|\not\partial\Gamma^{\leq 59}u_{k-1}\|_{L^2L^2(C_\tau^R)} + \|r^{-1}\Gamma^{\leq 60}u_{k-1}\|_{L^2L^2(C_\tau^R)}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int\int_{C_\tau^U} r|u_{k-1}||\Gamma^{\leq 60}u_{k-1}||\Gamma^{\leq 30}u_{k-1}||\partial\Gamma^{\leq 59}u_{k-1}|dxdt \\
&\quad\lesssim \left(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\leq 3}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\leq 2}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\right) \\
&\quad\times \left(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\leq 33}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\leq 32}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\right) \\
&\qquad\qquad\qquad\times \|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\leq 60}u_{k-1}\|_{L^2L^2(C_\tau^U)} \\
&\qquad\qquad\times \Big(\|\not\partial\Gamma^{\leq 59}u_{k-1}\|_{L^2L^2(C_\tau^U)} + \|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\leq 60}u_{k-1}\|_{L^2L^2(C_\tau^U)} \\
&\qquad\qquad\qquad\qquad + \|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\leq 59}u_{k-1})\|_{L^2L^2(C_\tau^U)}\Big),
\end{aligned}
$$

respectively. Thus, after summing over R, U, and τ, we have

$$
\begin{aligned}
&\int_0^T\int r|u_{k-1}||\Gamma^{\leq 60}u_{k-1}||\Gamma^{\leq 30}u_{k-1}||\partial\Gamma^{\leq 59}u_{k-1}|dxdt \lesssim (VI_{k-1}+V_{k-1})^3 VI_{k-1} \\
&\quad + \log\langle T\rangle(\log\langle T\rangle VIII_{k-1} + VII_{k-1})^2 \log\langle T\rangle VIII_{k-1}(V_{k-1} + \log\langle T\rangle VIII_{k-1} + VII_{k-1}). \quad (4.41)
\end{aligned}
$$

Continuing in this way, for the third term on the right of (4.39), we will show that

$$\int_0^T \int r|u_{k-1}||\Gamma^{\le 60}u_{k-1}|^2|\partial\Gamma^{\le 30}u_{k-1}|dxdt \lesssim (VI_{k-1}+V_{k-1})VI_{k-1}^2(IX_{k-1}+X_{k-1})$$
$$+\log\langle T\rangle(\log\langle T\rangle VIII_{k-1}+VII_{k-1})(\log\langle T\rangle VIII_{k-1})^2(\log\langle T\rangle XI_{k-1}+\log\langle T\rangle XII_{k-1}). \quad (4.42)$$

This follows from

$$\int\int_{C_\tau^R} r|u_{k-1}||\Gamma^{\le 60}u_{k-1}|^2|\partial\Gamma^{\le 30}u_{k-1}|dxdt$$
$$\lesssim \Big(\|r^{-1}\Gamma^{\le 3}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}+\|\bar{\partial}\Gamma^{\le 2}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big)\|r^{-1}\Gamma^{\le 60}u_{k-1}\|^2_{L^2L^2(C_\tau^R)}$$
$$\Big(\|\partial\Gamma^{\le 33}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}+R\|\bar{\partial}\partial\Gamma^{\le 32}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big)$$

and

$$\int\int_{C_\tau^U} r|u_{k-1}||\Gamma^{\le 60}u_{k-1}|^2|\partial\Gamma^{\le 30}u_{k-1}|dxdt$$
$$\lesssim \Big(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\le 3}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}+\|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\le 2}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)$$
$$\times\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\le 60}u_{k-1}\|^2_{L^2L^2(C_\tau^U)}$$
$$\times\Big(\tau^{-1/2}U^{1/2}\|\partial\Gamma^{\le 33}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}+\tau^{1/2}U^{1/2}\|\bar{\partial}\partial\Gamma^{\le 32}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)$$

where we used (1.14) in addition to the usual (3.16) and (3.17).

Finally, for the last term on the right of (4.39), one can show

$$\int\int_{C_\tau^R} r|u_{k-1}||\Gamma^{\le 60}u_{k-1}||\partial\Gamma^{\le 30}u_{k-1}||\partial\Gamma^{\le 60}u_{k-1}|dxdt$$
$$\lesssim \Big(\|r^{-1}\Gamma^{\le 3}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}+\|\bar{\partial}\Gamma^{\le 2}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big)\|r^{-1}\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^R)}$$
$$\times\Big(\|\partial\Gamma^{\le 33}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}+R\|\bar{\partial}\partial\Gamma^{\le 32}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big)R^{-1}\|\partial\Gamma^{\le 60}u_{k-1}\|_{L^2L^2(C_\tau^R)}$$

and

$$\int\int_{C_\tau^U} r|u_{k-1}||\Gamma^{\leq 60}u_{k-1}||\partial\Gamma^{\leq 30}u_{k-1}||\partial\Gamma^{\leq 60}u_{k-1}|dxdt$$
$$\lesssim \left(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\leq 3}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\leq 2}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\right)$$
$$\times \left(\tau^{-1/2}U^{1/2}\|\partial\Gamma^{\leq 33}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \tau^{1/2}U^{1/2}\|\bar{\partial}\partial\Gamma^{\leq 32}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}\right)$$
$$\times \|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\leq 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}U^{-1/2}\|\langle r\rangle^{-1/2}\partial\Gamma^{\leq 60}u_{k-1}\|_{L^2L^2(C_\tau^U)}.$$

By summing over $R \leq \tau/4$, $U \leq \tau/4$, and $\tau \leq T$ and using (4.8), (4.9), and (1.14), we find

$$\int_0^T\int r|u_{k-1}||\Gamma^{\leq 60}u_{k-1}||\partial\Gamma^{\leq 30}u_{k-1}||\partial\Gamma^{\leq 60}u_{k-1}|dxdt$$
$$\lesssim (VI_{k-1}+V_{k-1})VI_{k-1}(IX_{k-1}+X_{k-1})I_{k-1}$$
$$+ (\log\langle T\rangle VIII_{k-1}+VII_{k-1})(\log\langle T\rangle VIII_{k-1})(\log\langle T\rangle XI_{k-1}+\log\langle T\rangle XII_{k-1})(\log\langle T\rangle)^{1/2}I_{k-1}. \tag{4.43}$$

Together (4.39), (4.40), (4.41), (4.42), and (4.43) establish

$$\sup_{U\geq 1}\sup_{t\in[0,T]}\Big|\int_0^t\int re^{-\sigma_U(t-r)}a^{I,\alpha}_{JK}\omega_\alpha u^J_{k-1}\Gamma^{\leq 60}u^K_{k-1}$$
$$\times\left(\Box\Gamma^{\leq 60}u^I_k - a^{I,\beta}_{PQ}\omega_\beta u^P_{k-1}(\partial_t-\partial_r)\Gamma^{\leq 60}u^Q_{k-1}\right)dxdt\Big| \lesssim (\log\langle T\rangle)^5M^4_{k-1}. \tag{4.44}$$

To finish the proof of (4.37), we must consider

$$\int_0^t\int re^{-\sigma_U(t-r)}a^{I,\alpha}_{JK}\omega_\alpha u^J_{k-1}\Gamma^{\leq 60}u^K_{k-1}\left(a^{I,\beta}_{PQ}\omega_\beta u^P_{k-1}(\partial_t-\partial_r)\Gamma^{\leq 60}u^Q_{k-1}\right)dxdt.$$

Using the product rule, we may rewrite this term as

$$\frac{1}{2}\int_0^t\int re^{-\sigma_U(t-r)}u^J_{k-1}u^P_{k-1}(\partial_t-\partial_r)\left(a^{I,\alpha}_{JK}\omega_\alpha a^{I,\beta}_{PQ}\omega_\beta\Gamma^{\leq 60}u^K_{k-1}\Gamma^{\leq 60}u^Q_{k-1}\right)dxdt.$$

Integration by parts in $\partial_t - \partial_r$ then shows

$$\begin{aligned}
\int_0^t \int re^{-\sigma_U(t-r)} u_{k-1}^J u_{k-1}^P (\partial_t - \partial_r) \left(a_{JK}^{I,\alpha}\omega_\alpha a_{PQ}^{I,\beta}\omega_\beta \Gamma^{\le 60} u_{k-1}^K \Gamma^{\le 60} u_{k-1}^Q \right) dxdt \\
= \int re^{-\sigma_U(t-r)} u_{k-1}^J u_{k-1}^P a_{JK}^{I,\alpha}\omega_\alpha a_{PQ}^{I,\beta}\omega_\beta \Gamma^{\le 60} u_{k-1}^K \Gamma^{\le 60} u_{k-1}^Q dx\Big|_0^t \\
+ 3\int_0^t \int e^{-\sigma_U(t-r)} u_{k-1}^J u_{k-1}^P a_{JK}^{I,\alpha}\omega_\alpha a_{PQ}^{I,\beta}\omega_\beta \Gamma^{\le 60} u_{k-1}^K \Gamma^{\le 60} u_{k-1}^Q dxdt \\
+ 2\int_0^t \int re^{-\sigma_U(t-r)} \sigma_U'(t-r) u_{k-1}^J u_{k-1}^P \Gamma^{\le 60} u_{k-1}^K \Gamma^{\le 60} u_{k-1}^Q dxdt \\
- 2\int_0^t re^{-\sigma_U(t-r)} u_{k-1}^P (\partial_t - \partial_r) u_{k-1}^J a_{JK}^{I,\alpha}\omega_\alpha a_{PQ}^{I,\beta}\omega_\beta \Gamma^{\le 60} u_{k-1}^K \Gamma^{\le 60} u_{k-1}^Q dxdt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sup_{U\ge 1}\sup_{t\in[0,T]} \left| \int_0^t \int re^{-\sigma_U(t-r)} a_{JK}^{I,\alpha}\omega_\alpha u_{k-1}^J \Gamma^{\le 60} u_{k-1}^K \left(a_{PQ}^{I,\beta}\omega_\beta u_{k-1}^P (\partial_t - \partial_r) \Gamma^{\le 60} u_{k-1}^Q \right) dxdt \right| \\
\lesssim \sup_{t\in[0,T]} \int r|u_{k-1}|^2 |\Gamma^{\le 60} u_{k-1}|^2 dx + \int_0^T \int |u_{k-1}|^2 |\Gamma^{\le 60} u_{k-1}|^2 dxdt \\
+ \int_0^T \int r\langle t-r\rangle^{-1} |u_{k-1}|^2 |\Gamma^{\le 60} u_{k-1}|^2 dxdt + \int_0^T \int r|u_{k-1}||\partial u_{k-1}||\Gamma^{\le 60} u_{k-1}|^2 dxdt. \quad (4.45)
\end{aligned}$$

We will now show that each term on the right side of (4.45) is bounded by $(\log\langle T\rangle)^5 M_{k-1}^4$.

For the first term on the right side of (4.45), we fix $t \in [0,T]$ and take $\tau \approx t$. By writing

$$\begin{aligned}
\int r|u_{k-1}|^2 |\Gamma^{\le 60} u_{k-1}|^2 dx \\
\lesssim \sum_{R\le \tau/4} \int_{A_R} r|u_{k-1}|^2 |\Gamma^{\le 60} u_{k-1}|^2 dx + \sum_{U\le\tau/4} \int_{\langle t-r\rangle \approx U} r|u_{k-1}|^2 |\Gamma^{\le 60} u_{k-1}|^2 dx,
\end{aligned}$$

we may apply (3.16) and (3.17) to $|u_{k-1}|$. Thus,

$$\begin{aligned}
\int_{A_R} r|u_{k-1}|^2 |\Gamma^{\le 60} u_{k-1}|^2 dx \lesssim R\|\Gamma^{\le 60} u_{k-1}(t,\cdot)\|_{L^2(A_R)}^2 \\
\times \left(\tau^{-1/2} R^{-3/2} \|\Gamma^{\le 3} u_{k-1}\|_{L^2L^2(\tilde C_\tau^R)} + \tau^{-1/2} R^{-1/2} \|(\partial_t + \partial_r)\Gamma^{\le 2} u_{k-1}\|_{L^2L^2(\tilde C_\tau^R)} \right)^2
\end{aligned}$$

and

$$\int_{\langle t-r\rangle\approx U} r|u_{k-1}|^2|\Gamma^{\le 60}u_{k-1}|^2dx \lesssim \tau\|\Gamma^{\le 60}u_{k-1}(t,\cdot)\|^2_{L^2(\langle t-r\rangle\approx U)}$$
$$\times\Big(\tau^{-3/2}U^{-1/2}\|\Gamma^{\le 3}u_{k-1}\|_{L^2L^2(\tilde C^U_\tau)} + \tau^{-3/2}U^{-1/2}\|(\partial_t+\partial_r)(r\Gamma^{\le 2}u_{k-1})\|_{L^2L^2(\tilde C^U_\tau)}\Big)^2.$$

By rearranging weights, we see

$$\int_{A_R} r|u_{k-1}|^2|\Gamma^{\le 60}u_{k-1}|^2dx \lesssim \|r^{-1/2}\Gamma^{\le 60}u_{k-1}(t,\cdot)\|^2_{L^2(A_R)}$$
$$\times\Big(\|r^{-1}\Gamma^{\le 3}u_{k-1}\|_{L^2L^2(\tilde C^R_\tau)} + \|\bar\partial\Gamma^{\le 2}u_{k-1}\|_{L^2L^2(\tilde C^R_\tau)}\Big)^2.$$

Furthermore,

$$\int_{\langle t-r\rangle\approx U} r|u_{k-1}|^2|\Gamma^{\le 60}u_{k-1}|^2dx \lesssim \|r^{-1/2}\Gamma^{\le 60}u_{k-1}(t,\cdot)\|^2_{L^2(\langle t-r\rangle\approx U)}$$
$$\times\Big(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\le 3}u_{k-1}\|_{L^2L^2(\tilde C^U_\tau)} + \|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\le 2}u_{k-1})\|_{L^2L^2(\tilde C^U_\tau)}\Big)^2.$$

After summing and taking a supremum over $t\in[0,T]$, it follows that

$$\sup_{t\in[0,T]}\int r|u_{k-1}|^2|\Gamma^{\le 60}u_{k-1}|^2dx \lesssim IV^2_{k-1}(VI_{k-1}+V_{k-1})^2 + IV^2_{k-1}(\log\langle T\rangle VIII_{k-1}+VII_{k-1})^2. \quad (4.46)$$

For the second term on the right of (4.45), (3.16) and (3.17) give

$$\int\int_{C^R_\tau}|u_{k-1}|^2|\Gamma^{\le 60}u_{k-1}|^2dxdt$$
$$\lesssim\Big(\tau^{-1/2}R^{-3/2}\|\Gamma^{\le 3}u_{k-1}\|_{L^2L^2(\tilde C^R_\tau)} + \tau^{-1/2}R^{-1/2}\|(\partial_t+\partial_r)\Gamma^{\le 2}u_{k-1}\|_{L^2L^2(\tilde C^R_\tau)}\Big)^2$$
$$\times\|\Gamma^{\le 60}u_{k-1}\|^2_{L^2L^2(C^R_\tau)}$$

and respectively,

$$\int\int_{C_\tau^U} |u_{k-1}|^2 |\Gamma^{\leq 60} u_{k-1}|^2 dxdt$$
$$\lesssim \Big(\tau^{-3/2} U^{-1/2} \|\Gamma^{\leq 3} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \tau^{-3/2} U^{-1/2} \|(\partial_t + \partial_r)(r\Gamma^{\leq 2} u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)^2$$
$$\times \|\Gamma^{\leq 60} u_{k-1}\|^2_{L^2L^2(C_\tau^U)}.$$

It follows that

$$\int\int_{C_\tau^R} |u_{k-1}|^2 |\Gamma^{\leq 60} u_{k-1}|^2 dxdt$$
$$\lesssim \Big(\|r^{-1}\Gamma^{\leq 3} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + \|\partial\!\!\!/ \Gamma^{\leq 2} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big)^2 \|r^{-1}\Gamma^{\leq 60} u_{k-1}\|^2_{L^2L^2(C_\tau^R)}$$

and

$$\int\int_{C_\tau^U} |u_{k-1}|^2 |\Gamma^{\leq 60} u_{k-1}|^2 dxdt$$
$$\lesssim \Big(\|r^{-1/2}\langle t-r\rangle^{-1/2} \Gamma^{\leq 3} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t + \partial_r)(r\Gamma^{\leq 2} u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)^2$$
$$\times \|r^{-1}\Gamma^{\leq 60} u_{k-1}\|^2_{L^2L^2(C_\tau^U)}.$$

Using (1.20), we obtain

$$\int_0^T \int |u_{k-1}|^2 |\Gamma^{\leq 60} u_{k-1}|^2 dxdt \lesssim (VI_{k-1} + V_{k-1})^2 VI_{k-1}^2 + (\log\langle T\rangle VIII_{k-1} + VII_{k-1})^2 VI_{k-1}^2. \quad (4.47)$$

Following a nearly identical argument, we see for $R \leq \tau/4$,

$$\int\int_{C_\tau^R} r\langle t-r\rangle^{-1} |u_{k-1}|^2 |\Gamma^{\leq 60} u_{k-1}|^2 dxdt$$
$$\lesssim \Big(\|r^{-1}\Gamma^{\leq 3} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + \|\partial\!\!\!/ \Gamma^{\leq 2} u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\Big)^2 \|r^{-1}\Gamma^{\leq 60} u_{k-1}\|^2_{L^2L^2(C_\tau^R)}$$

and for $U \leq \tau/4$,

$$\int\int_{C_\tau^U} r\langle t-r\rangle^{-1}|u_{k-1}|^2|\Gamma^{\leq 60}u_{k-1}|^2 dxdt$$
$$\lesssim \Big(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\leq 3}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\leq 2}u_{k-1})\|_{L^2L^2(\tilde{C}_\tau^U)}\Big)^2$$
$$\times \|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\leq 60}u_{k-1}\|^2_{L^2L^2(C_\tau^U)}.$$

Thus the third term on the right of (4.45) is bounded using

$$\int_0^T\int r\langle t-r\rangle^{-1}|u_{k-1}|^2|\Gamma^{\leq 60}u_{k-1}|^2 dxdt \lesssim (VI_{k-1}+V_{k-1})^2 VI_{k-1}^2$$
$$+\log\langle T\rangle(\log\langle T\rangle VIII_{k-1}+VII_{k-1})^2(\log\langle T\rangle VIII_{k-1})^2. \quad (4.48)$$

Finally, the fourth term on the right of (4.45) is controlled using (4.42). In conjunction with (4.46), (4.47), and (4.48), this results in

$$\sup_{U\geq 1}\sup_{t\in[0,T]}\left|\int_0^t\int re^{-\sigma_U(t-r)}a^{I,\alpha}_{JK}\omega_\alpha u^J_{k-1}\Gamma^{\leq 60}u^K_{k-1}\left(a^{I,\beta}_{PQ}\omega_\beta u^P_{k-1}(\partial_t-\partial_r)\Gamma^{\leq 60}u^Q_{k-1}\right)dxdt\right|$$
$$\lesssim (\log\langle T\rangle)^5 M_{k-1}^4.$$

Along with (4.36), (4.38), and (4.44), this proves (4.37) and completes the proof of boundedness for terms III_k through $VIII_k$.

4.1.3 Term IX_k

For $R \leq \tau/4$, using (3.3), we can bound

$$\|\partial\Gamma^{\leq 50}u_k\|_{L^2L^2(\tilde{C}_\tau^R)} \lesssim \|\not\partial\Gamma^{\leq 50}u_k\|_{L^2L^2(\tilde{C}_\tau^R)} + \|r^{-1}\Gamma^{\leq 51}u_k\|_{L^2L^2(\tilde{C}_\tau^R)}.$$

Therefore, it immediately follows that

$$IX_k^2 \lesssim V_k^2 + VI_k^2. \quad (4.49)$$

As such, (4.13), (4.14), and (4.15) give the desired bound.

4.1.4 Term X_k

For $1 \leq R \leq \tau/4$, we may apply (3.7) so that

$$R\|\bar{\partial}\partial\Gamma^{\leq 40}u_k\|_{L^2L^2(\tilde{C}_\tau^R)} \lesssim \|\bar{\partial}\Gamma^{\leq 41}u_k\|_{L^2L^2(\tilde{C}_\tau^R)} + R\|\Box\Gamma^{\leq 40}u_k\|_{L^2L^2(\tilde{C}_\tau^R)} + \|\partial\Gamma^{\leq 40}u_k\|_{L^2L^2(\tilde{C}_\tau^R)}. \quad (4.50)$$

Upon summing the terms on the right side of (4.50), the first is controlled by V_k and the third is controlled by IX_k.

For the second term on the right of (4.50), unlike (4.16), we do not pay attention to which factors receive fewer vector fields. Additionally, when two derivatives are present, we may treat one as a vector field. Therefore, using (1.14) and (4.1), we note that

$$|\Box\Gamma^{\leq 40}u_k| \lesssim |\Gamma^{\leq 40}u_{k-1}||\partial\Gamma^{\leq 40}u_{k-1}|.$$

Applying (3.16) then gives

$$R\|\Box\Gamma^{\leq 40}u_k\|_{L^2L^2(\tilde{C}_\tau^R)} \lesssim \left(R^{-1}\|\Gamma^{\leq 43}u_{k-1}\|_{L^2L^2(\tilde{\tilde{C}}_\tau^R)} + \|\bar{\partial}\Gamma^{\leq 42}u_{k-1}\|_{L^2L^2(\tilde{\tilde{C}}_\tau^R)}\right) \times \|\partial\Gamma^{\leq 40}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}.$$

Note that when $R = \tau/4$, we are unable to cover $\tilde{\tilde{C}}_\tau^R$ by the regions $\tilde{C}_\tau^{\overline{R}}$ for any $\overline{R} \leq \tau/4$. Therefore, we bound such terms using the analogous terms over $\tilde{C}_\tau^U$ with $U = \tau/4$. After summing over $R \leq \tau/4$ and $\tau \leq T$, we conclude

$$X_k \lesssim V_k + (VI_{k-1} + V_{k-1} + \log\langle T\rangle VIII_{k-1} + VII_{k-1})IX_{k-1} + IX_k.$$

Therefore, by (4.13), (4.14), (4.15), and (4.49), it follows that

$$X_k^2 \leq (C_3\varepsilon)^2 + C(\log\langle T\rangle)^3 M_{k-1}^2 M_k + C(\log\langle T\rangle)^5 M_{k-1}^4$$

for a fixed constant C_3.

4.1.5 Term XI_k

In the case where $U = 1$, we first note that

$$\|\partial\Gamma^{\leq 50} u_k\|^2_{L^2L^2(\tilde{C}^{U=1}_\tau)} \lesssim \tau \|\partial\Gamma^{\leq 50} u_k\|^2_{L^\infty L^2}.$$

Furthermore, since we sum over dyadic values of τ,

$$\sum_{\tau \geq 4} \frac{1}{(\log\langle\tau\rangle)^2} < \infty.$$

Therefore,

$$\sum_{\tau \geq 4} \frac{1}{\tau(\log\langle\tau\rangle)^2} \|\partial\Gamma^{\leq 50} u_k\|^2_{L^2L^2(\tilde{C}^{U=1}_\tau)} \lesssim I_k^2. \tag{4.51}$$

For $1 < U \leq \tau/4$, we may apply Proposition 3.2 so that

$$\frac{U^{1/2}}{\tau^{1/2}\log\langle\tau\rangle} \|\partial\Gamma^{\leq 50} u_k\|_{L^2L^2(\tilde{C}^U_\tau)} \lesssim \|\cancel{\partial}\Gamma^{\leq 50} u_k\|_{L^2L^2(\tilde{C}^U_\tau)}$$
$$+ \|r^{-1/2}\langle t-r\rangle^{-1/2}(\log\langle r\rangle)^{-1}\Gamma^{\leq 51} u_k\|_{L^2L^2(\tilde{C}^U_\tau)} + \|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t + \partial_r)(r\Gamma^{\leq 50} u_k)\|_{L^2L^2(\tilde{C}^U_\tau)}.$$

By summing over $\tau \geq 4U$, we have

$$\sum_{\tau \geq 4U} \left(\frac{U^{1/2}}{\tau^{1/2}\log\langle\tau\rangle} \|\partial\Gamma^{\leq 50} u_k\|_{L^2L^2(\tilde{C}^U_\tau)} \right)^2 \lesssim V_k^2 + VIII_k^2 + VII_k^2. \tag{4.52}$$

Together (4.51) and (4.52) show

$$XI_k^2 \lesssim I_k^2 + V_k^2 + VII_k^2 + VIII_k^2, \tag{4.53}$$

which is controlled using (4.5), (4.13), and (4.15). This is the desired boundedness for term XI_k.

4.1.6 Term XII_k

As above, we will consider $U = 1$ and $U > 1$ separately. First, when $U = 1$, we write

$$\|\cancel{\partial}\partial\Gamma^{\leq 40} u_k\|_{L^2L^2(\tilde{C}^{U=1}_\tau)} \leq \|\cancel{\nabla}\partial\Gamma^{\leq 40} u_k\|_{L^2L^2(\tilde{C}^{U=1}_\tau)} + \|(\partial_t + \partial_r)\partial\Gamma^{\leq 40} u_k\|_{L^2L^2(\tilde{C}^{U=1}_\tau)}.$$

On $\tilde{C}_\tau^{U=1}$, we may simply treat ∂_r as a vector field and we have $\langle t-r\rangle^{-1} \lesssim 1$. Thus, applying (1.9) yields

$$\frac{\tau^{1/2}}{\log\langle\tau\rangle}\|\bar{\partial}\partial\Gamma^{\leq 40}u_k\|_{L^2L^2(\tilde{C}_\tau^{U=1})} \lesssim \|r^{-1/2}\langle t-r\rangle^{-1/2}(\log\langle r\rangle)^{-1}\Gamma^{\leq 42}u_k\|_{L^2L^2(\tilde{C}_\tau^{U=1})}$$
$$+ \|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\leq 41}u_k)\|_{L^2L^2(\tilde{C}_\tau^{U=1})}.$$

Therefore, by summing over $\tau \geq 4$, we see

$$\sum_{\tau\geq 4}\left(\frac{\tau^{1/2}}{\log\langle\tau\rangle}\|\bar{\partial}\partial\Gamma^{\leq 40}u_k\|_{L^2L^2(\tilde{C}_\tau^{U=1})}\right)^2 \lesssim VIII_k^2 + VII_k^2. \tag{4.54}$$

For $1 < U \leq \tau/4$, we may apply (3.8) so that

$$\frac{\tau^{1/2}U^{1/2}}{\log\langle\tau\rangle}\|\bar{\partial}\partial\Gamma^{\leq 40}u_k\|_{L^2L^2(\tilde{C}_\tau^U)} \lesssim \|\bar{\partial}\Gamma^{\leq 41}u_k\|_{L^2L^2(\tilde{C}_\tau^U)} + \frac{\tau^{1/2}U^{1/2}}{\log\langle\tau\rangle}\|\Box\Gamma^{\leq 40}u_k\|_{L^2L^2(\tilde{C}_\tau^U)}$$
$$+ \frac{U^{1/2}}{\tau^{1/2}\log\langle\tau\rangle}\|\partial\Gamma^{\leq 40}u_k\|_{L^2L^2(\tilde{C}_\tau^U)}. \tag{4.55}$$

Upon summing over $\tau \geq 4U$, the first and third terms on the right of (4.55) are controlled by terms V_k and XI_k, respectively.

In order to appropriately bound the second term on the right of (4.55), we use (1.14) and (4.1) so that

$$|\Box\Gamma^{\leq 40}u_k| \lesssim |\partial\Gamma^{\leq 40}u_{k-1}||\Gamma^{\leq 40}u_{k-1}|,$$

as we did in Section 4.1.4. Therefore, an application of (3.17) provides the bound

$$\frac{\tau^{1/2}U^{1/2}}{\log\langle\tau\rangle}\|\Box\Gamma^{\leq 40}u_k\|_{L^2L^2(\tilde{C}_\tau^U)} \lesssim \frac{U^{1/2}}{\tau^{1/2}\log\langle\tau\rangle}\|\partial\Gamma^{\leq 40}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}$$
$$\times \big(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\leq 43}u_{k-1}\|_{L^2L^2(\tilde{\tilde{C}}_\tau^U)} + \|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\leq 42}u_{k-1})\|_{L^2L^2(\tilde{\tilde{C}}_\tau^U)}\big).$$

Finally, summing over $\tau \geq 4U$ gives

$$\sum_{\tau\geq 4U}\left(\frac{\tau^{1/2}U^{1/2}}{\log\langle\tau\rangle}\|\bar{\partial}\partial\Gamma^{\leq 40}u_k\|_{L^2L^2(\tilde{C}_\tau^U)}\right)^2 \lesssim V_k^2 + XI_{k-1}^2(\log\langle T\rangle VIII_{k-1} + VII_{k-1})^2 + XI_k^2. \tag{4.56}$$

From (4.54), (4.56), and taking a supremum over $U \geq 1$, it immediately follows that

$$XII_k^2 \lesssim VIII_k^2 + VII_k^2 + V_k^2 + XI_{k-1}^2(\log\langle T\rangle VIII_{k-1} + VII_{k-1})^2 + XI_k^2.$$

Thus, (4.13), (4.14), (4.15), and (4.53) sufficiently bound term XII_k. This completes the proof of (4.4).

4.2 Proof of Convergence

In order to show that (u_k) is a convergent sequence, it suffices to show that (u_k) is Cauchy. To that end, we define

$$A[w] = \|\partial\Gamma^{\leq 20}w\|_{L^\infty L^2} + \|\Gamma^{\leq 20}w\|_{LE^1},$$

and for each $k \geq 1$, we set

$$A_k = A[u_k - u_{k-1}].$$

We will show for each $k \geq 1$, that

$$A_k \leq \frac{1}{2}A_{k-1}. \tag{4.57}$$

Assuming (4.57) holds, for each $k \geq j \geq 1$, it follows that

$$A[u_k - u_j] \leq \sum_{i=j+1}^{k} A_i \leq A_1 \sum_{i=j}^{k-1} \left(\frac{1}{2}\right)^i.$$

As $\sum_{i=1}^{\infty}(\frac{1}{2})^i$ is a convergent geometric series, this shows that (u_k) is Cauchy. It remains to prove (4.57).

4.2.1 A Hardy-type Inequality

In the proof of (4.57), the following Hardy-type inequality will be convenient.

Lemma 4.1. *Suppose $w \in C^1([0,T] \times \mathbb{R}^3)$ with support in $\{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 : r \leq t+1\}$. Then,*

$$\int_{\mathbb{R}^3} \frac{1}{(1+r)(t-r+2)^2} w^2\, dx \lesssim \int_{\mathbb{R}^3} \frac{1}{(1+r)r^2} w^2\, dx + \int_{\mathbb{R}^3} \frac{1}{1+r}(\partial_r w)^2\, dx. \tag{4.58}$$

Proof. First, we note that

$$\partial_r((t-r+2)^{-1}) = (t-r+2)^{-2}.$$

Thus, after converting to spherical coordinates and integrating by parts in r, we see

$$\int_{\mathbb{R}^3} \frac{1}{(1+r)(t-r+2)^2} w^2\,dx = -\int_{\mathbb{R}^3} \frac{(2+r)w^2}{(t-r+2)(1+r)^2 r}\,dx - 2\int_{\mathbb{R}^3} \frac{w\partial_r w}{(t-r+2)(1+r)}\,dx.$$

An application of the Schwarz inequality then gives

$$\int_{\mathbb{R}^3} \frac{1}{(1+r)(t-r+2)^2} w^2\,dx \lesssim \Big(\int_{\mathbb{R}^3} \frac{w^2}{(1+r)(t-r+2)^2}\,dx\Big)^{1/2} \times \left[\Big(\int_{\mathbb{R}^3} \frac{(2+r)^2 w^2}{(1+r)^3 r^2}\,dx\Big)^{1/2} + \Big(\int_{\mathbb{R}^3} \frac{(\partial_r w)^2}{1+r}\,dx\Big)^{1/2}\right]. \tag{4.59}$$

The desired result follows from dividing both sides by the first factor on the right of (4.59) and using the bound

$$\frac{(2+r)^2}{(1+r)^2} \leq 4 \tag{4.60}$$

for all $r \geq 0$.

□

4.2.2 Proof of (4.57)

For $k \geq 1$, we begin by applying the standard local energy estimate (2.1) to A_k so that

$$A_k^2 \lesssim \int_0^T \int |\Box\Gamma^{\leq 20}(u_k - u_{k-1})| \left(|\partial\Gamma^{\leq 20}(u_k - u_{k-1})| + \frac{|\Gamma^{\leq 20}(u_k - u_{k-1})|}{r}\right) dxdt. \tag{4.61}$$

Using (4.1), we note that

$$\Box(u_k - u_{k-1}) = a^{I,\alpha}_{JK}(u^J_{k-1} - u^J_{k-2})\partial_\alpha u^K_{k-1} + a^{I,\alpha}_{JK} u^J_{k-2}\partial_\alpha(u^K_{k-1} - u^K_{k-2}) + b^{I,\alpha\beta}_{JK}\partial_\alpha(u^J_{k-1} - u^J_{k-2})\partial_\beta u^K_{k-1} + b^{I,\alpha\beta}_{JK}\partial_\alpha u^J_{k-2}\partial_\beta(u^K_{k-1} - u^K_{k-2}).$$

As such, by (1.10) and (1.14), we can bound

$$|\Box\Gamma^{\leq 20}(u_k - u_{k-1})| \lesssim |\partial^{\leq 1}\Gamma^{\leq 20}(u_{k-1} - u_{k-2})||\partial\Gamma^{\leq 20}u_{k-1}| + |\partial^{\leq 1}\Gamma^{\leq 20}u_{k-2}||\partial\Gamma^{\leq 20}(u_{k-1} - u_{k-2})|, \tag{4.62}$$

where we do not pay attention to which terms are lower order. Using (1.20), it suffices to bound the integral in (4.61) over the regions C_τ^R and C_τ^U and sum over $R, U \le \tau/4$ and $\tau \le T$. We will do so using each term on the right of (4.62).

To that end, for the first term on the right of (4.62), applying (3.16), (1.14), and the Schwarz inequality gives

$$\begin{aligned}
&\int\int_{C_\tau^R} |\partial^{\le 1}\Gamma^{\le 20}(u_k - u_{k-1})||\partial\Gamma^{\le 20}u_{k-1}| \left(|\partial\Gamma^{\le 20}(u_k - u_{k-1})| + \frac{|\Gamma^{\le 20}(u_k - u_{k-1})|}{r}\right) dxdt \\
&\lesssim \|\langle r\rangle^{-3/2}\partial^{\le 1}\Gamma^{\le 20}(u_{k-1} - u_{k-2})\|_{L^2L^2(C_\tau^R)} \left(\|\partial\Gamma^{\le 23}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)} + R\|\not{\partial}\partial\Gamma^{\le 22}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^R)}\right) \\
&\qquad \times \left(\|\langle r\rangle^{-1/2}\partial\Gamma^{\le 20}(u_k - u_{k-1})\|_{L^2L^2(C_\tau^R)} + \|\langle r\rangle^{-1/2}r^{-1}\Gamma^{\le 20}(u_k - u_{k-1})\|_{L^2L^2(C_\tau^R)}\right). \quad (4.63)
\end{aligned}$$

Using (3.17) instead shows

$$\begin{aligned}
&\int\int_{C_\tau^U} |\partial^{\le 1}\Gamma^{\le 20}(u_k - u_{k-1})||\partial\Gamma^{\le 20}u_{k-1}| \left(|\partial\Gamma^{\le 20}(u_k - u_{k-1})| + \frac{|\Gamma^{\le 20}(u_k - u_{k-1})|}{r}\right) dxdt \\
&\qquad \lesssim \|\langle t-r\rangle^{-1}\langle r\rangle^{-1/2}\partial^{\le 1}\Gamma^{\le 20}(u_{k-1} - u_{k-2})\|_{L^2L^2(C_\tau^U)} \\
&\qquad \times \left(\tau^{-1/2}U^{1/2}\|\partial\Gamma^{\le 23}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)} + \tau^{1/2}U^{1/2}\|\not{\partial}\partial\Gamma^{\le 22}u_{k-1}\|_{L^2L^2(\tilde{C}_\tau^U)}\right) \\
&\qquad \times \left(\|\langle r\rangle^{-1/2}\partial\Gamma^{\le 20}(u_k - u_{k-1})\|_{L^2L^2(C_\tau^U)} + \|\langle r\rangle^{-1/2}r^{-1}\Gamma^{\le 20}(u_k - u_{k-1})\|_{L^2L^2(C_\tau^U)}\right). \quad (4.64)
\end{aligned}$$

To bound the first factor on the right of (4.64), we use Lemma 4.1 so that

$$\begin{aligned}
\|\langle t-r\rangle^{-1}\langle r\rangle^{-1/2}\partial^{\le 1}\Gamma^{\le 20}(u_{k-1} - u_{k-2})\|_{L^2L^2} &\lesssim \|\langle r\rangle^{-1/2}r^{-1}\Gamma^{\le 21}(u_{k-1} - u_{k-2})\|_{L^2L^2} \\
&\quad + \|\langle r\rangle^{-1/2}\partial\Gamma^{\le 21}(u_{k-1} - u_{k-2})\|_{L^2L^2}. \quad (4.65)
\end{aligned}$$

Such terms are bounded using (4.9).

Therefore, using (4.63), (4.64), (4.65), and (4.9), we may sum over $R, U \le \tau/4$ and $\tau \le T$ to see

$$\begin{aligned}
&\int_0^T\int |\partial^{\le 1}\Gamma^{\le 20}(u_k - u_{k-1})||\partial\Gamma^{\le 20}u_{k-1}| \left(|\partial\Gamma^{\le 20}(u_k - u_{k-1})| + \frac{|\Gamma^{\le 20}(u_k - u_{k-1})|}{r}\right) dxdt \\
&\qquad \lesssim (\log\langle T\rangle)^2 M_{k-1}A_{k-1}A_k. \quad (4.66)
\end{aligned}$$

Similar arguments for the second term on the right of (4.62) show

$$\int\int_{C_\tau^R} |\partial^{\le 1}\Gamma^{\le 20}u_{k-2}||\partial\Gamma^{\le 20}(u_{k-1}-u_{k-2})| \left(|\partial\Gamma^{\le 20}(u_k-u_{k-1})| + \frac{|\Gamma^{\le 20}(u_k-u_{k-1})|}{r}\right) dxdt$$
$$\lesssim \left(\|r^{-1}\Gamma^{\le 24}u_{k-2}\|_{L^2L^2(\tilde{C}_\tau^R)} + \|\not\partial\Gamma^{\le 23}u_{k-2}\|_{L^2L^2(\tilde{C}_\tau^R)}\right) \|\langle r\rangle^{-1/2}\partial\Gamma^{\le 20}(u_{k-1}-u_{k-2})\|_{L^2L^2(\tilde{C}_\tau^R)}$$
$$\times \left(\|\langle r\rangle^{-1/2}\partial\Gamma^{\le 20}(u_k-u_{k-1})\|_{L^2L^2(C_\tau^R)} + \|\langle r\rangle^{-1/2}r^{-1}\Gamma^{\le 20}(u_k-u_{k-1})\|_{L^2L^2(C_\tau^R)}\right)$$

and

$$\int\int_{C_\tau^U} |\partial^{\le 1}\Gamma^{\le 20}u_{k-2}||\partial\Gamma^{\le 20}(u_{k-1}-u_{k-2})| \left(|\partial\Gamma^{\le 20}(u_k-u_{k-1})| + \frac{|\Gamma^{\le 20}(u_k-u_{k-1})|}{r}\right) dxdt$$
$$\lesssim \left(\|r^{-1/2}\langle t-r\rangle^{-1/2}\Gamma^{\le 24}u_{k-2}\|_{L^2L^2(\tilde{C}_\tau^U)} + \|r^{-1/2}\langle t-r\rangle^{-1/2}(\partial_t+\partial_r)(r\Gamma^{\le 23}u_{k-2})\|_{L^2L^2(\tilde{C}_\tau^U)}\right)$$
$$\times \|\langle r\rangle^{-1/2}\partial\Gamma^{\le 20}(u_{k-1}-u_{k-2})\|_{L^2L^2(C_\tau^U)}$$
$$\times \left(\|\langle r\rangle^{-1/2}\partial\Gamma^{\le 20}(u_k-u_{k-1})\|_{L^2L^2(C_\tau^U)} + \|\langle r\rangle^{-1/2}\Gamma^{\le 20}(u_k-u_{k-1})\|_{L^2L^2(C_\tau^U)}\right).$$

By summing and once again using (4.9), we see that

$$\int_0^T\int |\partial^{\le 1}\Gamma^{\le 20}u_{k-2}||\partial\Gamma^{\le 20}(u_{k-1}-u_{k-2})| \left(|\partial\Gamma^{\le 20}(u_k-u_{k-1})| + \frac{|\Gamma^{\le 20}(u_k-u_{k-1})|}{r}\right) dxdt$$
$$\lesssim (\log\langle T\rangle)^2 M_{k-2}A_{k-1}A_k. \quad (4.67)$$

From (4.61), (4.62), (4.66), and (4.67), we conclude

$$A_k^2 \lesssim (\log\langle T\rangle)^2(M_{k-1}+M_{k-2})A_{k-1}A_k. \quad (4.68)$$

Subsequently, assuming $c \ll 1$, we use (1.6), bound M_{k-1} and M_{k-2} using (4.3), and divide both sides by A_k. It follows that

$$A_k^2 \lesssim c^4\varepsilon^{2/3}A_{k-1}^2.$$

Therefore, for $\varepsilon > 0$ sufficently small, (4.57) holds. This completes the proof of convergence for the sequence (u_k).

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