



Global existence for systems of quasilinear wave equations in $(1 + 4)$ -dimensions

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Abstract

Hörmander proved global existence of solutions for sufficiently small initial data for scalar wave equations in $(1 + 4)$ -dimensions of the form $\square u = Q(u, u', u'')$ where Q vanishes to second order and $(\partial_u^2 Q)(0, 0, 0) = 0$. Without the latter condition, only almost global existence may be guaranteed. The first author and Sogge considered the analog exterior to a star-shaped obstacle. Both results relied on writing the lowest order terms $u \partial_\alpha u = \frac{1}{2} \partial_\alpha u^2$ and as such do not immediately generalize to systems. The current study remedies such and extends both results to the case of multiple speed systems.

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1. Introduction

In [25], the authors study a quasilinear wave equation in four spatial dimensions whose non-linearity $Q(u, \partial u, \partial^2 u)$ vanishes to second order and is subject to the restriction

$$(\partial_u^2 Q)(0, 0, 0) = 0, \tag{1.1}$$

which has the effect of disallowing a term of the form u^2 . The global existence result of [8] is extended to equations exterior to star-shaped obstacles with Dirichlet boundary conditions. In [25], the first author inserted a careless comment that the techniques carry over with trivial change to the case of systems of equations. This, unfortunately, is not the case as the terms of

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the form $u\partial u$ were written as $\frac{1}{2}\partial u^2$ in order to apply [25, Proposition 4]. The current study seeks to remedy this by providing techniques that do suffice to show global existence for systems of equations of the same type. The original study [8] on $\mathbb{R} \times \mathbb{R}^4$ employed a similar technique as [25], and as such, the extension to systems appears to be new even in the $(1 + 4)$ -dimensional boundaryless case.

Let us more specifically introduce the problem at hand. We shall examine systems of quasi-linear equations of the form

$$\begin{cases} \square_{c_I} u^I = Q^I(u, u', u''), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^4 \setminus \mathcal{K}, \quad I = 1, 2, \dots, M, \\ u^I(t, \cdot)|_{\partial\mathcal{K}} = 0, \\ u^I(0, \cdot) = f^I, \quad \partial_t u^I(0, \cdot) = g^I \end{cases} \tag{1.2}$$

where $\square_{c_I} = \partial_t^2 - c_I^2 \Delta$ is the d'Alembertian with speed c_I . We shall, without loss of generality, assume that

$$0 < c_1 \leq c_2 \leq \dots \leq c_M. \tag{1.3}$$

Here $u = (u^1, \dots, u^M)$. We use u' and ∂u to denote the space-time gradient $(\partial_t u, \nabla_x u)$, but reserve $\nabla u = \nabla_x u$ to denote the spatial gradient. The obstacle \mathcal{K} is taken to be compact, to have smooth boundary, and to be star-shaped (with respect to the origin). By translating and scaling, without loss of generality, we may take $0 \in \mathcal{K} \subset \{|x| < 1\}$.

The nonlinear term Q is smooth in its arguments, vanishes to second order, and satisfies (1.1). Moreover, it has the form

$$Q^I(u, u', u'') = A^I(u, u') + B^{IJ,\alpha\beta}(u, u') \partial_\alpha \partial_\beta u^J. \tag{1.4}$$

Here A is taken to vanish to second order at $(u, u') = (0, 0)$, and B vanishes to first order at the origin. We also assume the symmetry condition

$$B^{IJ,\alpha\beta}(u, u') = B^{JI,\alpha\beta}(u, u') = B^{IJ,\beta\alpha}(u, u'), \quad 1 \leq I, J \leq M, 0 \leq \alpha, \beta \leq 4. \tag{1.5}$$

The above uses the convenient notation $\partial_0 = \partial_t$. Here and throughout this paper, we shall utilize the summation convention where repeated indices are summed. Greek indices α, β, γ are summed from 0 to the spatial dimension 4, while lower case Latin indices i, j, k are implicitly summed from 1 to 4. Repeated uppercase I, J, K will be implicitly summed from 1 to M . We shall reserve μ, ν , and σ to denote multiindices.

To solve (1.2), one must assume that the data satisfy some compatibility conditions. These are well known, and we shall only tersely describe them. A more detailed exposition is available in, e.g., [10]. We write $J_k u = \{\partial_x^\mu u : 0 \leq |\mu| \leq k\}$. For any formal H^m solution u , we can express $\partial_t^k u(0, \cdot) = \psi_k(J_k f, J_{k-1} g)$, $0 \leq k \leq m$ for some compatibility functions ψ_k . The compatibility condition of order m for data $(f, g) \in H^m \times H^{m-1}$ simply requires that ψ_k vanishes on $\partial\mathcal{K}$ for $0 \leq k \leq m - 1$. For $(f, g) \in C^\infty$, we say that the data satisfy the compatibility condition to infinite order if the above holds for all m .

Theorem 1.1. *Suppose $\mathcal{K} \subset \{x \in \mathbb{R}^4 : |x| < 1\}$ is star-shaped with respect to the origin and has smooth boundary. Suppose Q satisfies (1.1), (1.4), and (1.5). Assume that $(f, g) \in$*

$(C^\infty(\mathbb{R}^4 \setminus \mathcal{K}))^{2M}$ satisfy the compatibility conditions to infinite order, have components that are supported in $(\mathbb{R}^4 \setminus \mathcal{K}) \cap \{|x| < R_0\}$, and

$$\sum_{|\mu| \leq N} \|\nabla^\mu f\|_{L^2(\mathbb{R}^4 \setminus \mathcal{K})} + \sum_{|\mu| \leq N} \|\nabla^\mu g\|_{L^2(\mathbb{R}^4 \setminus \mathcal{K})} = \varepsilon. \tag{1.6}$$

Then if $N \geq 12$ and $\varepsilon > 0$ is sufficiently small, (1.2) has a unique global solution.

In the above theorem, we have taken the data to be compactly supported for convenience and for clarity of exposition. It is highly likely that the same result holds for sufficiently small and regular data that decay sufficiently fast at infinity. Moreover, the assumed regularity is not nearly sharp. The methods employed will not approach the sharp threshold.

Such quasilinear wave equations in $\mathbb{R}_+ \times \mathbb{R}^4$ whose nonlinearity depends only on derivatives of the solution were known to enjoy global existence for sufficiently small data [13]. Dependence on the solution rather than its derivatives does not mesh as well with the energy methods that are typically applied. In this case, [8] shows that without the hypothesis (1.1) almost global existence, which means that the lifespan grows exponentially as the size of the data shrinks, is what is possible. Moreover, [8] continues and demonstrates that (1.1) suffices to establish small data global existence for scalar equations. To the authors’ knowledge, however, the current result for systems is new even in the boundaryless context.

Following the work [9], which first demonstrated the utility of local energy estimates as a means to proving long time existence for wave equations in exterior domains, a number of studies tackled the existence of solutions to quasilinear wave equations with small initial data in exterior domains when the nonlinearity depends only on derivatives of the solution. See., e.g., [11], [19, 20], [21,22] for some of the most general results. Only having the local energy estimate for the flat wave equation, which fails to account for the geometric perturbations introduced by the quasilinear nonlinearities, requires additional techniques in order to control the highest order energies. When, however, the obstacle is assumed to be star-shaped, due to the existence of a local energy estimate for sufficiently small time-dependent perturbations of the wave equation, simpler techniques, which are more direct analogs of [9], are available. See [22,24].

Like in the boundaryless case, nonlinear dependence on the solution rather than only its derivatives complicates the situation. The works [3], [2], and [31] explore the general case outside of star-shaped obstacles and prove analogs of the lifespans in the boundaryless case of [16], [8], and [15], [18] respectively. See [5,6] for similar results in more general geometries. The subsequent work [25] examines scalar equations subject to (1.1) exterior to 4-dimensional star-shaped obstacles as in the boundaryless case of [8].

As mentioned previously, [8] and [25] rely on writing the terms of the form $u\partial u$ in divergence form $\frac{1}{2}\partial(u^2)$ for which better estimates are available. For systems of equations, however, it cannot be guaranteed that all such first order terms are in divergence form. The key new idea is to use a lemma from [27] that allows a derivative to be exchanged for additional decay within the light cone.

We work outside of star-shaped obstacles so that the techniques of [24] are available. As in [5,6], it is expected that this geometric condition could be relaxed significantly.

We shall rely on a variant of Klainerman’s method of invariant vector fields [12,13] as was adapted to exterior domains in [9,11], [21], and subsequent works. In particular, to avoid having vector fields with unbounded normal components on $\partial\mathcal{K}$, a restricted set of vector fields is em-

ployed. The generators of space-time translations, spatial rotations, and scaling will be used. We set

$$\Gamma = (\partial_0, \partial_1, \partial_2, \partial_3, \Omega_{12}, \Omega_{13}, \Omega_{14}, \Omega_{23}, \Omega_{24}, \Omega_{34}, S)$$

with

$$\Omega_{ij} = x_i \partial_j - x_j \partial_i, \quad S = t \partial_t + r \partial_r.$$

It is worth noting that $[\partial, \Gamma]u = \mathcal{O}(|\partial u|)$. We will abbreviate multi-index notation and, for any $N \in \mathbb{N}$, denote

$$\Gamma^{\leq N} u = \sum_{|\mu| \leq N} \Gamma^\mu u, \quad \partial^{\leq N} u = \sum_{|\mu| \leq N} \partial^\mu u.$$

We end this section by fixing additional notations. We will denote $\square u = (\square_{c_l} u^l)$. And we will fix $\beta(\rho)$ to be a smooth cutoff that is identically 1 for $\rho \leq 1$ and 0 for $\rho > 2$. Furthermore, we let $\beta_R(\rho) = \beta(\rho/R)$.

We shall use $S_t = [0, t] \times \mathbb{R}^4$ and $S_t^{\mathcal{K}} = [0, t] \times (\mathbb{R}^4 \setminus \mathcal{K})$ to denote space-time strips in Minkowski space and the exterior domain respectively. We shall be working in mixed norms. When two Lebesgue spaces appear, it is understood to be a space-time norm:

$$\|u\|_{L^p L^q} = \left(\int_0^t \left(\int_{\mathbb{R}^4 \setminus \mathcal{K}} |u(s, x)|^q dx \right)^{p/q} ds \right)^{1/p}.$$

When three occurrences appear, this indicates that the norms are in $t, r,$ and ω with the convention

$$\|u\|_{L^p L^q L^\theta} = \left(\int_0^t \left[\int_0^\infty \left(\int_{\mathbb{S}^3} |u(s, r\omega)|^\theta d\sigma(\omega) \right)^{q/\theta} r^3 dr \right]^{p/q} ds \right)^{1/p}.$$

Due to the appearance of cutoff functions in the sequel, we shall only need this latter norm in the boundaryless case, which is what is stated here. We shall use L_{2k}^p to indicate that the norm is restricted to an annulus (or a ball in the case of $k = 0$): $c2^k < \langle \cdot \rangle < C2^k$. The norm may be in the time or the spatial variables. The positive constants c, C will be permitted to change from line to line but may not depend on any important parameters in the problem.

This article is organized as follows. In the next section, the main linear estimates are collected. These are variants of energy and local energy estimates. The first ones are local energy estimates for the solution without a derivative and are obtained by appropriately dividing through by a derivative and using a variant of a Sobolev embedding. These are collected from [2], [7], and [25]. The second class of estimates apply to small, time-dependent perturbations of \square and, as stated, are from [24] but are heavily influenced by the preceding works [30], [23]. The third section includes the main decay estimate, which is from [13] and yields decay in $|x|$ at the cost of admissible vector fields. Here we gather an estimate of [27] that allows us to exchange a derivative for additional decay when sufficiently within the light cone. The final section is devoted to the proof of Theorem 1.1.

2. Energy and local energy estimates

In this section, the linear L^2 based estimates, which are variants of energy and local energy bounds, are collected. For the first several estimates, as we will be cutting away from the obstacle, it will suffice to consider bounds in Minkowski space with vanishing initial data. The basic uniform energy bound and local energy estimate states that solutions to the scalar equation

$$\begin{cases} \square_c w = F, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^4, \\ w(0, \cdot) = \partial_t w(0, \cdot) = 0 \end{cases} \tag{2.1}$$

satisfy

$$\|w'(t, \cdot)\|_{L^2(\mathbb{R}^4)} + \|\langle x \rangle^{-1/2-} w'\|_{L^2 L^2(S_t)} \lesssim \|F\|_{L^1 L^2(S_t) + \langle x \rangle^{-1/2-} L^2 L^2(S_t)}. \tag{2.2}$$

Here and throughout, these estimates are global in nature, and the implicit constants are independent of t . See, e.g., [26] for some history and more general results.

As the nonlinearity under consideration allows for dependence on the solution rather than only its derivatives, variants of the above shall be required where the solution is estimated rather than only its derivatives. To that end, we record the following previously established results. See, also, [4], [17], and [28] for some closely related estimates.

Proposition 2.1 ([2, Theorem 2.3], [7, Lemma 3.1]). *Let w be a smooth solution to (2.1) where $\text{supp } F(t, \cdot) \subset \{|x| > 1\}$ for all $t \geq 0$. Then, for $t > 0$,*

$$\|w(t, \cdot)\|_{L^2(\mathbb{R}^4)} \lesssim \|\langle x \rangle^{-1} F\|_{L^1 L^1 L^2(S_t)}, \tag{2.3}$$

and for $0 < \gamma < \frac{1}{2}$, we have

$$\|\langle x \rangle^{-\frac{1}{2}-\gamma} w\|_{L^2 L^2(S_t)} \lesssim \|\langle x \rangle^{-1-\gamma} F\|_{L^1 L^1 L^2(S_t)}. \tag{2.4}$$

The first estimate follows by dividing through by a derivative and applying a Sobolev-type estimate that is akin to, e.g., [29, (3.20a)]. Specifically, see the dual version [2, Lemma 2.2]. The second estimate is established by proving a variant of (2.2) that relates the power of the weight to the regularity (see [7, (3.6)]) and combining such with a trace lemma.

The next result will allow us to easily handle the commutators that will appear when we cut off away from the obstacle in order to apply Proposition 2.1. This is essentially [2, (2.12)], [25, Proposition 3].

Proposition 2.2. *Let \tilde{w} be a smooth solution to (2.1), and suppose that $F(t, x) = 0$ for $|x| > 2R_0$. Then,*

$$\|\tilde{w}(t, \cdot)\|_{L^2(\mathbb{R}^4)} + \|\langle x \rangle^{-1/2-} \tilde{w}\|_{L^2 L^2(S_t)} \lesssim \|F\|_{L^2 L^2(S_t)}. \tag{2.5}$$

Here, as above, the constant is independent of t , but it may depend on R_0 .

In order to prove this result, (2.2) is applied to $w = \Delta^{-1} \partial_j \tilde{w}$. Since the kernel of $\Delta^{-1} \partial_j$ is $\mathcal{O}(|x - y|^{-3})$, Young’s inequality completes the proof.

A final such result shows that better bounds are available on the solution when the nonlinearity is in divergence form. The following is based on ideas of [8].

Proposition 2.3 ([25, Proposition 4]). *Let v be a smooth solution to*

$$\begin{cases} \square v = a_\alpha \partial_\alpha G, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^4, \\ v(0, \cdot) = \partial_t v(0, \cdot) = 0, \end{cases} \tag{2.6}$$

where a_α are constants. Moreover, assume that $G(0, \cdot) \equiv 0$. Then

$$\| \langle x \rangle^{-1/2-\delta} v \|_{L^2 L^2(S_t)} + \| v(t, \cdot) \|_{L^2(\mathbb{R}^4)} \lesssim \int_0^t \| G(s, \cdot) \|_{L^2(\mathbb{R}^4)} ds. \tag{2.7}$$

Here if $\square v_1 = G$ with vanishing data, it is noted that $v = a_\alpha \partial_\alpha v_1$. The result then follows from (2.2).

Due to the quasilinear nature of the study, we shall require variants of (2.2) that account for the geometry introduced by the nonlinearity. In particular, we need a version of (2.2) that holds for small, time-dependent perturbations of \square . To that end, we define

$$(\square_h u)^I = (\partial_t^2 - c_I^2 \Delta) u^I + h^{IJ, \alpha\beta}(t, x) \partial_\alpha \partial_\beta u^J,$$

and we consider

$$\begin{cases} \square_h u = F + G, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^4 \setminus \mathcal{K}, \\ u|_{\partial\mathcal{K}} = 0, \\ u(0, \cdot) = f, \quad \partial_t u(0, \cdot) = g. \end{cases} \tag{2.8}$$

We assume that the perturbation satisfies the symmetry conditions

$$h^{IJ, \alpha\beta} = h^{JI, \alpha\beta} = h^{IJ, \beta\alpha} \tag{2.9}$$

and the smallness condition

$$|h| := \sum_{I, J=1}^M \sum_{\alpha, \beta=0}^4 |h^{IJ, \alpha\beta}| < \delta \ll 1. \tag{2.10}$$

We also set the notation

$$|\partial h| = \sum_{I, J=1}^M \sum_{\alpha, \beta, \gamma=0}^4 |\partial_\gamma h^{IJ, \alpha\beta}|.$$

Proposition 2.4 ([24, Theorem 2.4]). *Suppose \mathcal{K} , as above, satisfies $0 \in \mathcal{K} \subset \{x \in \mathbb{R}^4 : |x| < 1\}$, is star-shaped with respect to the origin, and has a smooth boundary. Assume that the $h^{J,\alpha\beta}$ satisfy (2.9) and (2.10) for $\delta > 0$ sufficiently small. If $G(s, \cdot)$ is supported where $|x| < 2$ for every $s \in [0, t]$, if f, g vanish for $|x| > R_0$, and if u is a smooth solution to (2.8) that vanishes for large $|x|$ for each $s \in [0, t]$, then*

$$\begin{aligned} & \| \langle x \rangle^{-1/2-\Gamma} u' \|_{L^2 L^2(S_t^{\mathcal{K}})} + \| \Gamma^{\leq N} u'(t, \cdot) \|_{L^2(\mathbb{R}^4 \setminus \mathcal{K})} \lesssim \| \nabla^{\leq N} u'(0, \cdot) \|_{L^2(\mathbb{R}^4 \setminus \mathcal{K})} \\ & + \int_0^t \| \Gamma^{\leq N} F(s, \cdot) \|_{L^2(\mathbb{R}^4 \setminus \mathcal{K})} ds + \int_0^t \| [h^{J,\alpha\beta}(s, \cdot) \partial_\alpha \partial_\beta, \Gamma^{\leq N}] u^J(s, \cdot) \|_{L^2(\mathbb{R}^4 \setminus \mathcal{K})} ds \\ & + \int_0^t \| \Gamma^{\leq N-1} \square u(s, \cdot) \|_{L^2(\mathbb{R}^4 \setminus \mathcal{K})} ds + \int_0^t \left\| \left(|\partial h(s, \cdot)| + \frac{|h(s, \cdot)|}{\langle x \rangle} \right) |\Gamma^{\leq N} \partial u(s, \cdot)| \right\|_{L^2(\mathbb{R}^4 \setminus \mathcal{K})} ds \\ & + \sum_{|\mu|+|\nu| \leq N} \int_0^t \| |\Gamma^\nu h(s, \cdot)| |\Gamma^\mu u'(s, \cdot)| \|_{L^2((\mathbb{R}^4 \setminus \mathcal{K}) \cap \{|x| < 1\})} ds + \| \Gamma^{\leq N} G \|_{L^2 L^2(S_t^{\mathcal{K}})} \\ & + \| \Gamma^{\leq N-1} \square u \|_{L^2 L^2(S_t^{\mathcal{K}})} + \sup_{s \in [0, t]} \| \Gamma^{\leq N-1} \square u(s, \cdot) \|_{L^2(\mathbb{R}^4 \setminus \mathcal{K})} \quad (2.11) \end{aligned}$$

for any $N \geq 0$ and $t \geq 0$.

When $N = 0$, (2.11) is proved by pairing \square_h with $\frac{r}{r+2j} \partial_r u + \frac{3}{2} \frac{1}{r+2j} u$ and integrating by parts. As ∂_t preserves the boundary conditions, the estimate when Γ is replaced by ∂_t follows immediately. An elliptic regularity argument then yields the estimate where the Γ are all ∂ . Finally, the vector fields can be broken into $\Omega_{ij} = (1 - \beta(|x|))\Omega_{ij} + \beta(|x|)\Omega_{ij}$ and $S = \left(t \partial_t + (1 - \beta(|x|))r \partial_r \right) + \beta(|x|)r \partial_r$. The bounds for any term involving $\beta(|x|)$ follow from the preceding step, while the $(1 - \beta(|x|))$ cutoffs guarantee preservation of the boundary conditions. Thus, the $N = 0$ result may be applied and the commutators are included in the G , which in turn is handled by the estimate involving only derivatives as vector fields.

The result (2.11) differs slightly in appearance from [24, Theorem 2.4]. Here we have only applied the Schwarz inequality and bootstrapped the portions coming from the multiplier. As we are not using a multiple speed null condition, there is no loss in the current setting in recording this simplified version.

3. Sobolev-type decay bounds

As was first noted in [9], local energy estimates allow us to establish long time existence using decay in $|x|$ rather than decay in t . The latter, as in the Klainerman-Sobolev bounds [13], often relies on more symmetries of the equation. This may be manifest through the introduction of additional vector fields, which may not mesh well with the existence of the boundary.

On the other hand, decay in $|x|$ can be readily established while only relying on the generators of translations and spatial rotations, which all have bounded coefficients on $\partial\mathcal{K}$. Indeed, we have the following.

Lemma 3.1 ([13, Proposition 1]). *For $w \in C^\infty(\mathbb{R}^4)$ and $j \geq 0$,*

$$\|w\|_{L_{2^j}^\infty} \lesssim 2^{-3j/2} \|\Gamma^{\leq 3} w\|_{L_{2^j}^2}. \tag{3.1}$$

To prove this, after localizing, one applies Sobolev embeddings on $\mathbb{R} \times \mathbb{S}^3$. The result follows upon adjusting the volume element to match that of \mathbb{R}^4 in polar coordinates. Due to tails from the cutoff functions, the annulus on the right must be larger than that on the left, but our notation allows for such.

One of the key new ideas in this article when compared with [25] is the following lemma from [27, Lemma 3.11], which allows us to obtain further decay when the solution is differentiated. It will be convenient to have a version of this estimate that holds for small, time-dependent perturbations of \square , which is what we provide.

Lemma 3.2. *Assume that (1.3) holds. Moreover, assume that (2.9) and (2.10) hold for $\delta > 0$ sufficiently small. Then, for any $1 \ll R \leq c_1 T/8$, we have*

$$\begin{aligned} \|\partial w\|_{L^2 L^2([T, 2T] \times \{|x| \in [R, 2R]\})} &\lesssim R^{-1} \|S^{\leq 1} w\|_{L^2 L^2([T/2, 4T] \times \{|x| \in [R/2, 4R]\})} \\ &+ \|\partial h|w\|_{L^2 L^2([T/2, 4T] \times \{|x| \in [R/2, 4R]\})} + R \|\square_h w\|_{L^2 L^2([T/2, 4T] \times \{|x| \in [R/2, 4R]\})}. \end{aligned} \tag{3.2}$$

Proof. We begin with the pointwise estimate

$$|\nabla v|^2 \leq \frac{1}{(ct - r)^2} (Sv)^2 + \frac{1}{c} \frac{t}{ct - r} [c^2 |\nabla v|^2 - (\partial_t v)^2], \quad r < ct.$$

We now fix $\chi \in C^\infty(\mathbb{R} \times \mathbb{R})$ that is equal to 1 on $[1, 2] \times [1, 2]$ and supported in $[1/2, 4] \times [1/2, 4]$. Applying the above estimate to each component of $w = (w^I)$, multiplying by $\chi_{T,R}(t, x) = \chi(t/c_1 T, |x|/R)$, and integrating one obtains

$$\begin{aligned} \iint \chi_{T,R} |\nabla w|^2 dx dt &\lesssim \iint \frac{\chi_{T,R}}{T^2} |Sw|^2 dx dt \\ &+ \sum_{I=1}^M \iint \chi_{T,R} \frac{t}{c_I(c_I t - r)} [c_I^2 |\nabla w^I|^2 - (\partial_t w^I)^2] dx dt. \end{aligned}$$

As $\partial_t v = \frac{1}{t} Sv - \frac{r}{t} \partial_r v$, the above generalizes to

$$\begin{aligned} \iint \chi_{T,R} |\partial w|^2 dx dt &\lesssim \iint \frac{\chi_{T,R}}{T^2} |Sw|^2 dx dt \\ &+ \sum_{I=1}^M \iint \chi_{T,R} \frac{t}{c_I(c_I t - r)} [c_I^2 |\nabla w^I|^2 - (\partial_t w^I)^2] dx dt. \end{aligned}$$

Integrating by parts, however, yields

$$\begin{aligned} & \sum_{l=1}^M \iint \chi_{T,R} \frac{t}{c_I(c_I t - r)} [c_I^2 |\nabla w^l|^2 - (\partial_t w)^2] dx dt \\ &= \sum_{l=1}^M \iint \square_h w^l \chi_{T,R} \frac{t}{c_I(c_I t - r)} w^l dx dt - \frac{1}{2} \sum_{l=1}^M \iint \square_{c_I} \left(\chi_{T,R} \frac{t}{c_I(c_I t - r)} \right) (w^l)^2 dx dt \\ & \quad + \iint \chi_{T,R} \frac{t}{c_I(c_I t - r)} \left[\partial_\alpha h^{IJ,\alpha\beta} \partial_\beta w^J w^I + h^{IJ,\alpha\beta} \partial_\beta w^J \partial_\alpha w^I \right] dx dx \\ & \quad + \iint \partial_\alpha \left(\chi_{T,R} \frac{t}{c_I(c_I t - r)} \right) h^{IJ,\alpha\beta} \partial_\beta w^J w^I dx dt. \end{aligned}$$

The lemma follows from the two preceding equations as long as δ is sufficiently small so that the $h^{IJ,\alpha\beta} \partial_\beta w^J \partial_\alpha w^I$ term may be bootstrapped. \square

We end this section with a quick version of a Hardy inequality. While this is rather standard (see, e.g., [1]), we provide a proof to illustrate that one may rely only on the star-shapedness and not require the vanishing of the solution on the boundary.

Lemma 3.3. *Let \mathcal{K} be a star-shaped with respect to the origin, and suppose $v \in C^1(\mathbb{R}^4 \setminus \mathcal{K})$ vanishes at infinity. Then*

$$\|v/r\|_{L^2(\mathbb{R}^4 \setminus \mathcal{K})} \lesssim \|\nabla v\|_{L^2(\mathbb{R}^4 \setminus \mathcal{K})}.$$

Proof. Integrating the identity

$$\partial_i \left(\frac{x_i}{r^2} v^2 \right) = 2 \frac{v^2}{r^2} + 2 \frac{v}{r} \partial_r v$$

over $\mathbb{R}^4 \setminus \mathcal{K}$, we obtain

$$0 \geq - \int_{\partial \mathcal{K}} \langle x, n \rangle \frac{v^2}{r^2} d\sigma = 2 \|v/r\|_{L^2(\mathbb{R}^4 \setminus \mathcal{K})}^2 + 2 \int \frac{v}{r} \partial_r v dx.$$

Here n is the outward unit normal to \mathcal{K} . An application of the Schwarz inequality completes the proof. \square

4. Proof of Theorem 1.1

Since this is a small data result, higher order nonlinear terms are better behaved. Thus, for simplicity of exposition, we shall truncate Q to second order, which will allow us to assume

$$Q^I(u, u', u'') = a_{IJK}^\alpha u^J \partial_\alpha u^K + b_{IJK}^{\alpha\beta} \partial_\alpha u^J \partial_\beta u^K + A_{IJK}^{\alpha\beta} u^J \partial_\alpha \partial_\beta u^K + B_{IJK}^{\alpha\beta\gamma} \partial_\alpha u^J \partial_\beta \partial_\gamma u^K. \tag{4.1}$$

The constants $A_{IJK}^{\alpha\beta}, B_{IJK}^{\alpha\beta\gamma}$ are subject to

$$A_{IJK}^{\alpha\beta} = A_{IJK}^{\beta\alpha} = A_{KJI}^{\alpha\beta}, \quad B_{IJK}^{\alpha\beta\gamma} = B_{IJK}^{\alpha\gamma\beta} = B_{KJI}^{\alpha\beta\gamma}.$$

No u^2 term appears above due to (1.1).

4.1. Preliminaries

Here we will prove a couple of lemmas that will be useful for closing our iteration. Recall that we have fixed $R_0 > 1$ so that \mathcal{K} and the supports of the data are contained in $\{|x| < R_0\}$. Consider solutions u to

$$\begin{cases} \square_{c_I} u^I = a_{IJK}^\alpha \left[\phi^J \partial_\alpha \tilde{u}^K + \tilde{u}^J \partial_\alpha \psi_1^K \right] + b_{IJK}^{\alpha\beta} \left[\partial_\alpha \phi^J \partial_\beta \tilde{u}^K + \partial_\alpha \tilde{u}^J \partial_\beta \psi_1^K \right] \\ \quad + A_{IJK}^{\alpha\beta} \left[\phi^J \partial_\alpha \partial_\beta u^K + \tilde{u}^J \partial_\alpha \partial_\beta \psi_2^K \right] + B_{IJK}^{\alpha\beta\gamma} \left[\partial_\alpha \phi^J \partial_\beta \partial_\gamma u^K + \partial_\alpha \tilde{u}^J \partial_\beta \partial_\gamma \psi_2^K \right], \\ u^I(t, \cdot)|_{\partial\mathcal{K}} = 0, \\ u(0, \cdot) = f, \quad \partial_t u(0, \cdot) = g, \quad \text{supp}(f, g) \subseteq \{|x| \leq R_0\} \cap (\mathbb{R}^4 \setminus \mathcal{K}). \end{cases} \tag{4.2}$$

Here, $I = 1, 2, \dots, M$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^4 \setminus \mathcal{K}$, and $\phi, \psi_j, \tilde{u} \in (C^\infty(\mathbb{R}_+ \times \mathbb{R}^4 \setminus \mathcal{K}))^M$. The initial data are moreover assumed to be smooth and to satisfy the compatibility conditions to infinite order.

We shall set

$$\begin{aligned} M_N[u](T) = \sup_{t \in [0, T]} \left[(1+t)^{-\delta} \|\Gamma^{\leq N} u(t, \cdot)\|_{L^2(\mathbb{R}^4 \setminus \mathcal{K})} + \|\Gamma^{\leq N} u'(t, \cdot)\|_{L^2(\mathbb{R}^4 \setminus \mathcal{K})} \right] \\ + \|\langle x \rangle^{-3/4} \Gamma^{\leq N} u\|_{L^2 L^2(S_T^{\mathcal{K}})} + \|\langle x \rangle^{-3/4} \Gamma^{\leq N} u'\|_{L^2 L^2(S_T^{\mathcal{K}})} \end{aligned} \tag{4.3}$$

for some $\delta \geq 0$ sufficiently small.

The first lemma controls the local energy portions of $M_N[u]$.

Lemma 4.1. *For $u \in (C^\infty(\mathbb{R}_+ \times \mathbb{R}^4 \setminus \mathcal{K}))^M$ solving (4.2) and any N , we have*

$$\begin{aligned} \sup_{t \in [0, T]} \|\Gamma^{\leq N} u'(t, \cdot)\|_{L^2(\mathbb{R}^4 \setminus \mathcal{K})} + \|\langle x \rangle^{-3/4} \Gamma^{\leq N} u'\|_{L^2 L^2(S_T^{\mathcal{K}})} \\ + \sup_{t \in [0, T]} \|\beta_{2R_0}(|x|) \Gamma^{\leq N} u(t, \cdot)\|_{L^2(\mathbb{R}^4 \setminus \mathcal{K})} + \|\beta_{2R_0}(|x|) \Gamma^{\leq N} u\|_{L^2 L^2(S_T^{\mathcal{K}})} \\ \lesssim \|\nabla^{\leq N} u'(0, \cdot)\|_{L^2} + M_{N/2+4}[\phi] \left(M_N[\tilde{u}] + M_N[u] \right) \\ + \left(M_{N/2+3}[\tilde{u}] + M_{N/2+4}[u] \right) M_N[\phi] + \left(M_{N/2+3}[\psi_1] + M_{N/2+4}[\psi_2] \right) M_N[\tilde{u}] \\ + M_{N/2+3}[\tilde{u}] \left(M_N[\psi_1] + M_{N+1}[\psi_2] \right) \end{aligned} \tag{4.4}$$

provided $|\partial^{\leq 1} \phi| < \tilde{\delta}$ for some $\tilde{\delta} > 0$ sufficiently small.

The second lemma controls those portions of $M_N[u]$ that do not contain a derivative and are away from the boundary.

Lemma 4.2. For $u \in (C^\infty(\mathbb{R}_+ \times \mathbb{R}^4 \setminus \mathcal{K}))^M$ solving (4.2) and any N and any $t > 0$, we have

$$\begin{aligned}
 (1+t)^{-\delta} & \| (1 - \beta_{R_0}(|x|)) \Gamma^{\leq N} u(t, \cdot) \|_{L^2(\mathbb{R}^4 \setminus \mathcal{K})} + \| \langle x \rangle^{-3/4} (1 - \beta_{R_0}(|x|)) \Gamma^{\leq N} u \|_{L^2 L^2(S_t^{\mathcal{K}})} \\
 & \lesssim \| \nabla^{\leq N} u'(0, \cdot) \|_{L^2} + \left(\left[1 + M_{N/2+2}[\phi] \right] M_{N/2+3}[\tilde{u}] + M_{N/2+3}[u] \right) M_N[\phi] \\
 & + M_{N/2+4}[\phi] \left(\left[1 + M_{N/2+4}[\phi] + M_{N/2+3}[\psi_1] + M_{N/2+4}[\psi_2] \right] M_N[\tilde{u}] + M_N[u] \right) \\
 & + \left(M_{N/2+4}[\psi_1] + M_{N/2+4}[\psi_2] \right) M_N[\tilde{u}] + M_{N/2+3}[u] \left[1 + M_{N/2+2}[\phi] \right] \left(M_N[\psi_1] + M_{N+1}[\psi_2] \right) \\
 & + M_{N/2+2}[\phi] \| \langle x \rangle^{3/4} \Gamma^{\leq N-1} \square \tilde{u} \|_{L^2 L^2} + M_N[\phi] \| \langle x \rangle^{3/4} \Gamma^{\leq N/2+2} \square \tilde{u} \|_{L^2 L^2} \\
 & + \sum_{j=1,2} M_{N/2+2}[\tilde{u}] \| \langle x \rangle^{3/4} \Gamma^{\leq N} \square \psi_j \|_{L^2 L^2} + \sum_{j=1,2} M_N[\tilde{u}] \| \langle x \rangle^{3/4} \Gamma^{\leq N/2+3} \square \psi_j \|_{L^2 L^2} \\
 & + M_N[\tilde{u}] \| \langle x \rangle^{3/4} \Gamma^{\leq N/2+2} \square \phi \|_{L^2 L^2} + M_N[u] \| \langle x \rangle^{3/4} \Gamma^{\leq N/2+2} \square \phi \|_{L^2 L^2} \\
 & + M_N[\phi] \| \langle x \rangle^{3/4} \Gamma^{\leq N/2+3} \square u \|_{L^2 L^2} \tag{4.5}
 \end{aligned}$$

provided $|\partial^{\leq 1} \phi| < \tilde{\delta}$ for some $\tilde{\delta} > 0$ sufficiently small.

We shall delay the proofs of the lemmas and will instead first demonstrate that the lemmas can be used to complete the proof of Theorem 1.1. We will return to the proofs of the lemmas in the last two subsections, which will then complete the proof of the main result.

4.2. Proof of Theorem 1.1 assuming Lemma 4.1 and Lemma 4.2

We shall solve (1.2) via iteration. To that end, we let $u_0 \equiv 0$ and define u_n for $n \geq 1$ to solve

$$\begin{cases} \square_{c_l} u_n^I = Q^I(u_{n-1}, u'_{n-1}, u''_n), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^4 \setminus \mathcal{K}, \quad I = 1, 2, \dots, M \\ u_n^I(t, \cdot)|_{\partial \mathcal{K}} = 0, & t \geq 0, \\ u_n^I(0, \cdot) = f^I, & \partial_t u_n^I(0, \cdot) = g^I. \end{cases} \tag{4.6}$$

We shall begin by showing that the sequence $(M_N[u_n])_{n \in \mathbb{N}}$ is bounded for N sufficiently large, and we shall then subsequently use this to show that the sequence (u_n) is Cauchy and thus converges.

Boundedness: Our first goal is to argue inductively to show that there is a universal constant C_0 so that

$$M_N[u_n](T) \leq 10C_0 \varepsilon \tag{4.7}$$

for all $n \geq 0$ and all $T > 0$ provided that N is sufficiently large.

We first show that $M_N[u_1](T) \leq C_0 \varepsilon$. Noticing that u_1 solves (4.2) with $\phi, \psi_j, \tilde{u} \equiv 0$, this base case follows immediately from (4.4), (4.5), and (1.6).

We now assume that (4.7) holds for u_l for $l = 1, 2, \dots, n - 1$ and use the lemmas to establish the same for u_n . To do so, we note that u_n solves (4.2) with $\phi = \tilde{u} = u_{n-1}$ and $\psi_j \equiv 0$. We also note that

$$\begin{aligned}
 |\Gamma^{\leq N-1} \square u_{n-1}| &\lesssim |\Gamma^{\leq (N-1)/2} \partial^{\leq 1} u_{n-2}| |\Gamma^{\leq N-1} \partial u_{n-2}| \\
 &\quad + |\Gamma^{\leq (N-1)/2} \partial u_{n-2}| |\Gamma^{\leq N-1} \partial^{\leq 1} u_{n-2}| + |\Gamma^{\leq (N-1)/2} \partial^{\leq 1} u_{n-2}| |\Gamma^{\leq N} \partial u_{n-1}| \\
 &\quad + |\Gamma^{\leq (N-1)/2+1} \partial u_{n-1}| |\Gamma^{\leq N-1} \partial^{\leq 1} u_{n-2}|.
 \end{aligned}$$

Thus, by (3.1), we have

$$\begin{aligned}
 \|\langle x \rangle^{3/4} \Gamma^{\leq N-1} \square u_{n-1}\|_{L^2 L^2} &\lesssim \|\langle x \rangle^{-3/4} \Gamma^{\leq (N-1)/2+3} \partial^{\leq 1} u_{n-2}\|_{L^2 L^2} \|\Gamma^{\leq N-1} \partial u_{n-2}\|_{L^\infty L^2} \\
 &\quad + \|\Gamma^{\leq (N-1)/2+3} \partial u_{n-2}\|_{L^\infty L^2} \|\langle x \rangle^{-3/4} \Gamma^{\leq N-1} \partial^{\leq 1} u_{n-2}\|_{L^2 L^2} \\
 &\quad + \|\langle x \rangle^{-3/4} \Gamma^{\leq (N-1)/2+3} \partial^{\leq 1} u_{n-2}\|_{L^2 L^2} \|\Gamma^{\leq N} \partial u_{n-1}\|_{L^\infty L^2} \\
 &\quad + \|\Gamma^{\leq (N-1)/2+4} \partial u_{n-1}\|_{L^\infty L^2} \|\langle x \rangle^{-3/4} \Gamma^{\leq N-1} \partial^{\leq 1} u_{n-1}\|_{L^2 L^2}, \tag{4.8}
 \end{aligned}$$

which gives that this is

$$\begin{aligned}
 \mathcal{O}\left(M_{(N-1)/2+3}[u_{n-2}]M_{N-1}[u_{n-2}] + M_{(N-1)/2+3}[u_{n-2}]M_N[u_{n-1}] \right. \\
 \left. + M_{(N-1)/2+4}[u_{n-1}]M_{N-1}[u_{n-2}]\right).
 \end{aligned}$$

Since $N \geq 12$, we have that $(N/2) + 2 \leq N - 1$, so this same bound may be applied for $\|\langle x \rangle^{3/4} \Gamma^{\leq (N/2)+2} \square u_{n-1}\|_{L^2 L^2}$. Quite similarly, we have

$$\begin{aligned}
 \|\langle x \rangle^{3/4} \Gamma^{\leq N-1} \square u_n\|_{L^2 L^2} &= \mathcal{O}\left(M_{(N-1)/2+3}[u_{n-1}]M_{N-1}[u_{n-1}] \right. \\
 &\quad \left. + M_{(N-1)/2+3}[u_{n-1}]M_N[u_n] + M_{(N-1)/2+4}[u_n]M_{N-1}[u_{n-1}]\right). \tag{4.9}
 \end{aligned}$$

Assuming that $N \geq 12$ so that $(N/2) + 4 \leq N$, (4.4), (4.5), (1.6), and the inductive hypothesis show that

$$M_N[u_n] \leq C_0 \varepsilon + C^2(1 + C\varepsilon)\varepsilon^2 + C\varepsilon(1 + C\varepsilon)M_N[u_n],$$

which implies (4.7) so long as ε is sufficiently small.

Convergence: To complete the proof, we will show that

$$M_{N-1}[u_n - u_{n-1}](T) \leq \left(\frac{1}{2}\right)^{n-1} M_{N-1}[u_1](T) \tag{4.10}$$

for every n and all $T > 0$. This will imply that the sequence is Cauchy and, thus, convergent. Per standard arguments, the limiting value is our desired solution.

As the $n = 1$ case is trivial, we shall first show the base case

$$M_{N-1}[u_2 - u_1](T) \leq \frac{1}{2} M_{N-1}[u_1](T). \tag{4.11}$$

We shall then subsequently show that

$$M_{N-1}[u_n - u_{n-1}](T) \leq \frac{1}{4}M_{N-1}[u_{n-1} - u_{n-2}](T) + \frac{1}{8}M_{N-1}[u_{n-2} - u_{n-3}](T), \quad n \geq 3. \tag{4.12}$$

Then (4.10) follows from a straightforward argument relying on strong induction, which completes the proof modulo the proofs of Lemma 4.1 and Lemma 4.2.

Proof of (4.11). Observe that $u = u_2 - u_1$ solves (4.2) with vanishing data and with $\phi = \tilde{u} = \psi_2 = u_1$ and $\psi_1 \equiv 0$. Then (4.4), (4.5), and (4.7) give

$$M_{N-1}[u_2 - u_1] \leq C\varepsilon \left(M_{N-1}[u_1] + M_{N-1}[u_2 - u_1] \right) + C\varepsilon \|\langle x \rangle^{3/4} \Gamma^{\leq N-1} \square u_1\|_{L^2 L^2} + \|\langle x \rangle^{3/4} \Gamma^{\leq N-1} \square u_1\|_{L^2 L^2} M_{N-1}[u_2 - u_1] + M_{N-1}[u_1] \|\langle x \rangle^{3/4} \Gamma^{\leq N-2} \square (u_2 - u_1)\|_{L^2 L^2}$$

since $N \geq 12$ and $0 < \varepsilon \ll 1$. Note, however, that $\square u_1 = 0$.

We record that

$$|\Gamma^{\leq N-2} \square (u_2 - u_1)| \lesssim |\Gamma^{\leq N/2} \partial^{\leq 1} u_1| |\Gamma^{\leq N-1} \partial u_1| + |\Gamma^{\leq N/2} \partial u_1| |\Gamma^{\leq N-2} u_1| + |\Gamma^{\leq N/2-1} \partial^{\leq 1} u_1| |\Gamma^{\leq N-1} \partial (u_2 - u_1)| + |\Gamma^{\leq N/2} \partial (u_2 - u_1)| |\Gamma^{\leq N-2} \partial^{\leq 1} u_1|$$

since $N/2 + 4 \leq N - 2$ as $N \geq 12$. Then, by (3.1), we see that

$$\|\langle x \rangle^{3/4} \Gamma^{\leq N-2} \square (u_2 - u_1)\|_{L^2 L^2} \lesssim \|\langle x \rangle^{-3/4} \Gamma^{\leq N/2+3} \partial^{\leq 1} u_1\|_{L^2 L^2} \|\Gamma^{\leq N-1} \partial u_1\|_{L^\infty L^2} + \|\Gamma^{\leq N/2+3} \partial u_1\|_{L^\infty L^2} \|\langle x \rangle^{-3/4} \Gamma^{\leq N-2} u_1\|_{L^2 L^2} + \|\langle x \rangle^{-3/4} \Gamma^{\leq N/2+2} \partial^{\leq 1} u_1\|_{L^2 L^2} \|\Gamma^{\leq N-1} \partial (u_2 - u_1)\|_{L^\infty L^2} + \|\Gamma^{\leq N/2+3} \partial (u_2 - u_1)\|_{L^\infty L^2} \|\langle x \rangle^{-3/4} \Gamma^{\leq N-2} \partial^{\leq 1} u_1\|_{L^2 L^2}.$$

It then follows from (4.7) that this is $\mathcal{O}(\varepsilon^2 + \varepsilon M_{N-1}[u_2 - u_1])$. Plugging this into the above completes the proof provided ε is sufficiently small. \square

Proof of (4.12). Similar to the above, we note that $u = u_n - u_{n-1}$ solves (4.2) with $f, g \equiv 0$, $\phi = u_{n-1}$, $\tilde{u} = u_{n-1} - u_{n-2}$, $\psi_1 = u_{n-2}$, and $\psi_2 = u_{n-1}$. Thus, (4.4), (4.5), and (4.7) give

$$M_{N-1}[u_n - u_{n-1}] \leq C\varepsilon M_{N-1}[u_{n-1} - u_{n-2}] + C\varepsilon M_{N-1}[u_n - u_{n-1}] + C M_{N-1}[u_{n-1} - u_{n-2}] \sum_{j=1,2} \|\langle x \rangle^{3/4} \Gamma^{\leq N-1} \square u_{n-j}\|_{L^2 L^2} + M_{N-1}[u_n - u_{n-1}] \|\langle x \rangle^{3/4} \Gamma^{\leq N-2} \square u_{n-1}\|_{L^2 L^2} + C\varepsilon \|\langle x \rangle^{3/4} \Gamma^{\leq N-2} \square (u_{n-1} - u_{n-2})\|_{L^2 L^2} + C\varepsilon \|\langle x \rangle^{3/4} \Gamma^{\leq N-2} \square (u_n - u_{n-1})\|_{L^2 L^2} \tag{4.13}$$

provided $N \geq 12$. Notice that there is a loss of regularity associated to the ψ_2 piece, which requires the boundedness in a space with an extra degree of regularity. This is characteristic of quasilinear problems.

By (4.8), (4.9), and (4.7), the third and fourth terms in the right side of (4.12) are controlled by the first two terms. It only remains to show

$$\|\langle x \rangle^{3/4} \Gamma^{\leq N-2} \square(u_j - u_{j-1})\|_{L^2 L^2} \leq CM_{N-1}[u_{j-1} - u_{j-2}] + CM_{N-1}[u_j - u_{j-1}] \quad (4.14)$$

for $j = n - 1$ or $j = n$.

To this end, we calculate

$$\begin{aligned} |\Gamma^{\leq N-2} \square(u_j - u_{j-1})| &\lesssim |\Gamma^{\leq N/2-1} \partial^{\leq 1} u_{j-1}| |\Gamma^{\leq N-2} \partial(u_{j-1} - u_{j-2})| \\ &+ |\Gamma^{\leq N/2-1} \partial(u_{j-1} - u_{j-2})| |\Gamma^{\leq N-2} \partial^{\leq 1} u_{j-1}| + |\Gamma^{\leq N/2-1} \partial^{\leq 1} (u_{j-1} - u_{j-2})| |\Gamma^{\leq N-2} \partial u_{j-2}| \\ &\quad + |\Gamma^{\leq N/2-1} \partial u_{j-2}| |\Gamma^{\leq N-2} \partial^{\leq 1} (u_{j-1} - u_{j-2})| \\ &\quad + |\Gamma^{N/2-1} \partial^{\leq 1} u_{j-1}| |\Gamma^{\leq N-1} \partial(u_j - u_{j-1})| + |\Gamma^{\leq N/2} \partial(u_j - u_{j-1})| |\Gamma^{\leq N-2} \partial^{\leq 1} u_{j-1}| \\ &\quad + |\Gamma^{\leq N/2-1} \partial^{\leq 1} (u_{j-1} - u_{j-2})| |\Gamma^{\leq N-1} \partial u_{j-1}| + |\Gamma^{\leq N/2} \partial u_{j-1}| |\Gamma^{\leq N-2} \partial^{\leq 1} (u_{j-1} - u_{j-2})|. \end{aligned}$$

Applying (3.1), we obtain

$$\begin{aligned} \|\langle x \rangle^{3/4} \Gamma^{\leq N-2} \square(u_j - u_{j-1})\|_{L^2 L^2} &\lesssim \|\langle x \rangle^{-3/4} \Gamma^{\leq N/2+2} \partial^{\leq 1} u_{j-1}\|_{L^2 L^2} \|\Gamma^{\leq N-2} \partial(u_{j-1} - u_{j-2})\|_{L^\infty L^2} \\ &+ \|\Gamma^{\leq N/2+2} \partial(u_{j-1} - u_{j-2})\|_{L^\infty L^2} \|\langle x \rangle^{-3/4} \Gamma^{\leq N-2} \partial^{\leq 1} u_{j-1}\|_{L^2 L^2} \\ &+ \|\langle x \rangle^{-3/4} \Gamma^{\leq N/2+2} \partial^{\leq 1} (u_{j-1} - u_{j-2})\|_{L^2 L^2} \|\Gamma^{\leq N-2} \partial u_{j-2}\|_{L^\infty L^2} \\ &+ \|\Gamma^{\leq N/2+2} \partial u_{j-2}\|_{L^\infty L^2} \|\langle x \rangle^{-3/4} \Gamma^{\leq N-2} \partial^{\leq 1} (u_{j-1} - u_{j-2})\|_{L^2 L^2} \\ &\quad + \|\langle x \rangle^{-3/4} \Gamma^{N/2+2} \partial^{\leq 1} u_{j-1}\|_{L^2 L^2} \|\Gamma^{\leq N-1} \partial(u_j - u_{j-1})\|_{L^\infty L^2} \\ &\quad + \|\Gamma^{\leq N/2+3} \partial(u_j - u_{j-1})\|_{L^\infty L^2} \|\langle x \rangle^{-3/4} \Gamma^{\leq N-2} \partial^{\leq 1} u_{j-1}\|_{L^2 L^2} \\ &\quad + \|\langle x \rangle^{-3/4} \Gamma^{\leq N/2+2} \partial^{\leq 1} (u_{j-1} - u_{j-2})\|_{L^2 L^2} \|\Gamma^{\leq N-1} \partial u_{j-1}\|_{L^\infty L^2} \\ &\quad + \|\Gamma^{\leq N/2+3} \partial u_{j-1}\|_{L^\infty L^2} \|\langle x \rangle^{-3/4} \Gamma^{\leq N-2} \partial^{\leq 1} (u_{j-1} - u_{j-2})\|_{L^2 L^2}. \end{aligned}$$

From this, (4.14) follows immediately from (4.7). \square

We now complete the proof of Theorem 1.1 by proving Lemma 4.1 and Lemma 4.2.

4.3. Proof of Lemma 4.1

We first note that the Dirichlet boundary conditions allow us to control the last two terms in the left side of (4.4) by the first two terms. Here, the constant depends on R_0 , but this is harmless. Indeed, using the fact that ∂_α , Ω_{ij} , and $r \partial_r$ have bounded coefficients on the support of β_{2R_0} , we have

$$\begin{aligned} \|\Gamma^{\leq N} u(t, \cdot)\|_{L^2((\mathbb{R}^4 \setminus \mathcal{K}) \cap \{|x| < 2R_0\})} &\lesssim \|\Gamma^{\leq N-1} u'(t, \cdot)\|_{L^2((\mathbb{R}^4 \setminus \mathcal{K}) \cap \{|x| < 2R_0\})} \\ &\quad + \|(t \partial_t)^{\leq N} u(t, \cdot)\|_{L^2((\mathbb{R}^4 \setminus \mathcal{K}) \cap \{|x| < 2R_0\})}. \end{aligned}$$

For the last term, we may use the fact that $t \partial_t$ preserves the Dirichlet boundary conditions and integrate off of the boundary to see that

$$\|(t \partial_t)^{\leq N} u(t, \cdot)\|_{L^2((\mathbb{R}^4 \setminus \mathcal{K}) \cap \{|x| < 2R_0\})} \lesssim \|\Gamma^{\leq N} u'(t, \cdot)\|_{L^2((\mathbb{R}^4 \setminus \mathcal{K}) \cap \{|x| < 2R_0\})}.$$

To bound the first two terms in (4.4), we set $h^{IK, \alpha\beta} = -A_{IJK}^{\alpha\beta} \phi^J - B_{IJK}^{\gamma\alpha\beta} \partial_\gamma \phi^J$ and first note that

$$\begin{aligned} |\Gamma^{\leq N} \square_h u| &+ |[h^{IK, \alpha\beta} \partial_\alpha \partial_\beta, \Gamma^{\leq N}] u^K| + |\Gamma^{\leq N-1} \square u| \\ &\lesssim |\Gamma^{\leq N/2+1} \phi| |\Gamma^{\leq N} \partial \tilde{u}| + |\Gamma^{\leq N/2} \partial \tilde{u}| |\Gamma^{\leq N} \partial^{\leq 1} \phi| \\ &+ |\Gamma^{\leq N/2} \partial \psi_1| |\Gamma^{\leq N} \partial^{\leq 1} \tilde{u}| + |\Gamma^{\leq N/2} \partial^{\leq 1} \tilde{u}| |\Gamma^{\leq N} \partial \psi_1| \\ &+ |\Gamma^{\leq N/2+1} \phi| |\Gamma^{\leq N} \partial u| + |\Gamma^{\leq N/2+1} \partial u| |\Gamma^{\leq N} \partial^{\leq 1} \phi| \\ &+ |\Gamma^{\leq N/2} \partial^{\leq 1} \tilde{u}| |\Gamma^{\leq N+1} \partial \psi_2| + |\Gamma^{N/2+1} \partial \psi_2| |\Gamma^{\leq N} \partial^{\leq 1} \tilde{u}|. \end{aligned} \tag{4.15}$$

Due to (2.11), it suffices to control each of these terms in $L^1 L^2$, $L^2 L^2$, and $L^\infty L^2$.

To control the terms in $L^1 L^2$, we argue as in [9] and note that (3.1) and the Cauchy-Schwarz inequality give

$$\begin{aligned} \int_0^T \|vw\|_{L^2} ds &\leq \sum_{j \geq 0} \int_0^T \|vw\|_{L^2_j} ds \lesssim \sum_{j \geq 0} \int_0^T 2^{-3j/2} \|\Gamma^{\leq 3} v\|_{L^2_j} \|w\|_{L^2_j} ds \\ &\lesssim \sum_{j \geq 0} \|\langle x \rangle^{-3/4} \Gamma^{\leq 3} v\|_{L^2 L^2_j} \|\langle x \rangle^{-3/4} w\|_{L^2 L^2_j} \\ &\lesssim \|\langle x \rangle^{-3/4} \Gamma^{\leq 3} v\|_{L^2 L^2} \|\langle x \rangle^{-3/4} w\|_{L^2 L^2}. \end{aligned}$$

Applying this to each term in (4.15) where v is chosen to be the lower order term, we see that the $L^1 L^2$ -norm of (4.15) is controlled by a constant times

$$\begin{aligned} M_{N/2+4}[\phi] &\left(M_N[\tilde{u}] + M_N[u] \right) + \left(M_{N/2+3}[\tilde{u}] + M_{N/2+4}[u] \right) M_N[\phi] \\ &+ \left(M_{N/2+3}[\psi_1] + M_{N/2+4}[\psi_2] \right) M_N[\tilde{u}] + M_{N/2+3}[\tilde{u}] \left(M_N[\psi_1] + M_{N+1}[\psi_2] \right). \end{aligned} \tag{4.16}$$

We argue similarly to control the terms in $L^2 L^2$. Here (3.1) gives an excess of decay. We keep the L^2_t norm on the factor that may not contain a derivative so there is no loss of decay as is associated with the first term in (4.3). When each term in the right side of (4.15) is measured in $L^2 L^2$, we see that it is

$$\begin{aligned} &\lesssim \|\langle x \rangle^{-3/4} \Gamma^{\leq N/2+4} \phi\|_{L^2 L^2} \left(\|\Gamma^{\leq N} \partial \tilde{u}\|_{L^\infty L^2} + \|\Gamma^{\leq N} \partial u\|_{L^\infty L^2} \right) \\ &\quad + \left(\|\Gamma^{\leq N/2+3} \partial \tilde{u}\|_{L^\infty L^2} + \|\Gamma^{\leq N/2+4} \partial u\|_{L^\infty L^2} \right) \|\langle x \rangle^{-3/4} \Gamma^{\leq N} \partial^{\leq 1} \phi\|_{L^2 L^2} \\ &\quad + \left(\|\Gamma^{\leq N/2+3} \partial \psi_1\|_{L^\infty L^2} + \|\Gamma^{N/2+4} \partial \psi_2\|_{L^\infty L^2} \right) \|\langle x \rangle^{-3/4} \Gamma^{\leq N} \partial^{\leq 1} \tilde{u}\|_{L^2 L^2} \\ &\quad + \|\langle x \rangle^{-3/4} \Gamma^{\leq N/2+3} \partial^{\leq 1} \tilde{u}\|_{L^2 L^2} \left(\|\Gamma^{\leq N} \partial \psi_1\|_{L^\infty L^2} + \|\Gamma^{\leq N+1} \partial \psi_2\|_{L^\infty L^2} \right), \end{aligned}$$

which is in turn controlled by (4.16).

Finally, when measuring each term in $L^\infty L^2$, we apply (3.1) and pair a portion of the decay with the term that may not include a derivative as this allows for an application of a Hardy inequality to prevent having to handle the growth in t that is coupled to the first term in (4.3). Thus, when the right side of (4.15) is in $L^\infty L^2$, we have that it is

$$\begin{aligned} &\lesssim \|\langle x \rangle^{-1} \Gamma^{\leq N/2+4} \phi\|_{L^\infty L^2} \left(\|\Gamma^{\leq N} \partial \tilde{u}\|_{L^\infty L^2} + \|\Gamma^{\leq N} \partial u\|_{L^\infty L^2} \right) \\ &\quad + \left(\|\Gamma^{\leq N/2+3} \partial \tilde{u}\|_{L^\infty L^2} + \|\Gamma^{\leq N/2+4} \partial u\|_{L^\infty L^2} \right) \|\langle x \rangle^{-1} \Gamma^{\leq N} \partial^{\leq 1} \phi\|_{L^\infty L^2} \\ &\quad + \left(\|\Gamma^{\leq N/2+3} \partial \psi_1\|_{L^\infty L^2} + \|\Gamma^{N/2+4} \partial \psi_2\|_{L^\infty L^2} \right) \|\langle x \rangle^{-1} \Gamma^{\leq N} \partial^{\leq 1} \tilde{u}\|_{L^\infty L^2} \\ &\quad + \|\langle x \rangle^{-1} \Gamma^{\leq N/2+3} \partial^{\leq 1} \tilde{u}\|_{L^\infty L^2} \left(\|\Gamma^{\leq N} \partial \psi_1\|_{L^\infty L^2} + \|\Gamma^{\leq N+1} \partial \psi_2\|_{L^\infty L^2} \right). \end{aligned}$$

An application of Lemma 3.3 when the weighted terms do not include ∂ shows that these terms are also bounded by (4.16), which completes the proof. \square

4.4. Proof of Lemma 4.2

For $|\mu| \leq N$, we note that $(1 - \beta_{R_0}(|x|))\Gamma^\mu u^I$ solves the boundaryless equation

$$\begin{aligned} \square_{c_I} (1 - \beta_{R_0}(|x|))\Gamma^\mu u^I &= c_I^2 [\Delta, \beta_{R_0}(|x|)]\Gamma^\mu u^I + (1 - \beta_{R_0}(|x|))[\square_{c_I}, \Gamma^\mu] u^I \\ &\quad + (1 - \beta_{R_0}(|x|))\Gamma^\mu \square_{c_I} u^I \end{aligned}$$

with vanishing initial data due to the support conditions on f, g . By using (3.2), extra decay is available for terms involving derivatives, but it comes at the cost of a vector field. Thus, care is required when the highest number of vector fields lands on the term with derivatives. Similar care is required to prevent having a term with too many vector fields resulting from the full number of vector fields landing on a factor with two derivatives. To handle this, such terms are broken into a piece where the nonlinearity is in divergence form, for which we have (2.7), and a piece where the trouble is avoided.

To this end, for any $|\mu| \leq N$, we write

$$\begin{aligned} &(1 - \beta_{R_0}(|x|))\Gamma^\mu (a_{IJK}^\alpha \phi^J \partial_\alpha \tilde{u}^K) \\ &= \left[(1 - \beta_{R_0}(|x|))\Gamma^\mu (a_{IJK}^\alpha \phi^J \partial_\alpha \tilde{u}^K) - (1 - \beta_{R_0}(|x|))a_{IJK}^\alpha \phi^J \partial_\alpha (\Gamma^\mu \tilde{u}^K) \right] \\ &\quad - (1 - \beta_{R_0}(|x|))a_{IJK}^\alpha \partial_\alpha \phi^J \Gamma^\mu \tilde{u}^K + \partial_\alpha \left[(1 - \beta_{R_0}(|x|))a_{IJK}^\alpha \phi^J \Gamma^\mu \tilde{u}^K \right] \end{aligned}$$

$$\begin{aligned}
 & + \beta'_{R_0}(|x|) \frac{x_k}{r} a^k_{IJK} \phi^J \Gamma^\mu \tilde{u}^K \\
 = & \mathcal{O}\left(|\Gamma^{\leq N/2} \phi| |\Gamma^{\leq N-1} \partial \tilde{u}| + |\Gamma^{\leq N/2} \partial \tilde{u}| |\Gamma^{\leq N} \phi| + |\partial \phi| |\Gamma^{\leq N} \tilde{u}|\right) \\
 & + \partial_\alpha \left[(1 - \beta_{R_0}(|x|)) a^\alpha_{IJK} \phi^J \Gamma^\mu \tilde{u}^K \right] + \beta'_{R_0}(|x|) \frac{x_k}{r} a^k_{IJK} \phi^J \Gamma^\mu \tilde{u}^K.
 \end{aligned}$$

The terms involving ψ_j allow for a loss of regularity. As such, less care is required here, and we simply note that

$$(1 - \beta_{R_0}(|x|)) \Gamma^\mu (a^\alpha_{IJK} \tilde{u}^J \partial_\alpha \psi_1^K) = \mathcal{O}\left(|\Gamma^{\leq N/2} \tilde{u}| |\Gamma^{\leq N} \partial \psi_1| + |\Gamma^{\leq N/2} \partial \psi_1| |\Gamma^{\leq N} \tilde{u}|\right).$$

As

$$\begin{aligned}
 & (1 - \beta_{R_0}(|x|)) \Gamma^\mu (b^{\alpha\beta}_{IJK} \partial_\alpha \phi^J \partial_\beta \tilde{u}^K + \partial_\alpha \tilde{u}^J \partial_\beta \psi_1^K) \\
 & = \mathcal{O}\left(\left[|\Gamma^{\leq N/2} \partial \phi| + |\Gamma^{\leq N/2} \partial \psi_1|\right] |\Gamma^{\leq N} \partial \tilde{u}| + |\Gamma^{\leq N/2} \partial \tilde{u}| \left[|\Gamma^{\leq N} \partial \phi| + |\Gamma^{\leq N} \partial \psi_1|\right]\right),
 \end{aligned}$$

no modification is needed for this term. However, for the remaining pieces, we similarly arrange as follows:

$$\begin{aligned}
 & (1 - \beta_{R_0}(|x|)) \Gamma^\mu (A^{\alpha\beta}_{IJK} \phi^J \partial_\alpha \partial_\beta u^K) \\
 & = \left[(1 - \beta_{R_0}(|x|)) \Gamma^\mu (A^{\alpha\beta}_{IJK} \phi^J \partial_\alpha \partial_\beta u^K) - (1 - \beta_{R_0}(|x|)) A^{\alpha\beta}_{IJK} \phi^J \partial_\alpha \partial_\beta \Gamma^\mu u^K \right] \\
 & \quad - (1 - \beta_{R_0}(|x|)) A^{\alpha\beta}_{IJK} \partial_\alpha \phi^J \partial_\beta \Gamma^\mu u^K + \partial_\alpha \left[(1 - \beta_{R_0}(|x|)) A^{\alpha\beta}_{IJK} \phi^J \partial_\beta \Gamma^\mu u^K \right] \\
 & \quad + \beta'_{R_0}(|x|) \frac{x_k}{r} A^{k\beta}_{IJK} \phi^J \partial_\beta \Gamma^\mu u^K \\
 & = \mathcal{O}\left(|\Gamma^{\leq N/2} \phi| |\Gamma^{\leq N-1} \partial^2 u| + |\Gamma^{\leq N/2+1} \partial u| |\Gamma^{\leq N} \phi| + |\partial \phi| |\Gamma^{\leq N} \partial u|\right) \\
 & \quad + \partial_\alpha \left[(1 - \beta_{R_0}(|x|)) A^{\alpha\beta}_{IJK} \phi^J \partial_\beta \Gamma^\mu u^K \right] + \beta'_{R_0}(|x|) \frac{x_k}{r} A^{k\beta}_{IJK} \phi^J \partial_\beta \Gamma^\mu u^K
 \end{aligned}$$

and

$$\begin{aligned}
 & (1 - \beta_{R_0}(|x|)) \Gamma^\mu (B^{\alpha\beta\gamma}_{IJK} \partial_\gamma \phi^J \partial_\alpha \partial_\beta u^K) \\
 & = \left[(1 - \beta_{R_0}(|x|)) \Gamma^\mu (B^{\alpha\beta\gamma}_{IJK} \partial_\gamma \phi^J \partial_\alpha \partial_\beta u^K) - (1 - \beta_{R_0}(|x|)) B^{\alpha\beta\gamma}_{IJK} \partial_\gamma \phi^J \partial_\alpha \partial_\beta \Gamma^\mu u^K \right] \\
 & \quad - (1 - \beta_{R_0}(|x|)) B^{\alpha\beta\gamma}_{IJK} \partial_\alpha \partial_\gamma \phi^J \partial_\beta \Gamma^\mu u^K + \partial_\alpha \left[(1 - \beta_{R_0}(|x|)) B^{\alpha\beta\gamma}_{IJK} \partial_\gamma \phi^J \partial_\beta \Gamma^\mu u^K \right] \\
 & \quad + \beta'_{R_0}(|x|) \frac{x_k}{r} B^{k\beta\gamma}_{IJK} \partial_\gamma \phi^J \partial_\beta \Gamma^\mu u^K \\
 & = \mathcal{O}\left(|\Gamma^{\leq N/2} \partial \phi| |\Gamma^{\leq N} \partial u| + |\Gamma^{\leq N/2+1} \partial u| |\Gamma^{\leq N} \partial \phi|\right) \\
 & \quad + \partial_\alpha \left[(1 - \beta_{R_0}(|x|)) B^{\alpha\beta\gamma}_{IJK} \partial_\gamma \phi^J \partial_\beta \Gamma^\mu u^K \right] + \beta'_{R_0}(|x|) \frac{x_k}{r} B^{k\beta\gamma}_{IJK} \partial_\gamma \phi^J \partial_\beta \Gamma^\mu u^K.
 \end{aligned}$$

We similarly have these last two rearrangements when (ϕ, u) is replaced by (\tilde{u}, ψ_2) .

We write $(1 - \beta_{R_0}(|x|))\Gamma^\mu u^I = \tilde{w} + w + v$ where each has vanishing Cauchy data and

$$\begin{aligned} \square_{C_I} \tilde{w} &= c_I^2 [\Delta, \beta_{R_0}(|x|)]\Gamma^\mu u^I + \beta'_{R_0}(|x|) \frac{x_k}{r} a_{IJK}^k \phi^J \Gamma^\mu \tilde{u}^K \\ &\quad + \beta'_{R_0}(|x|) \frac{x_k}{r} A_{IJK}^{k\beta} \phi^J \partial_\beta \Gamma^\mu u^K + \beta'_{R_0}(|x|) \frac{x_k}{r} B_{IJK}^{k\beta\gamma} \partial_\gamma \phi^J \partial_\beta \Gamma^\mu u^K \\ &\quad + \beta'_{R_0}(|x|) \frac{x_k}{r} A_{IJK}^{k\beta} \tilde{u}^J \partial_\beta \Gamma^\mu \psi_2^K + \beta'_{R_0}(|x|) \frac{x_k}{r} B_{IJK}^{k\beta\gamma} \partial_\gamma \tilde{u}^J \partial_\beta \Gamma^\mu \psi_2^K \end{aligned}$$

and

$$\begin{aligned} \square_{C_I} v &= \partial_\alpha \left[(1 - \beta_{R_0}(|x|)) a_{IJK}^\alpha \phi^J \Gamma^\mu \tilde{u}^K \right] + \partial_\alpha \left[(1 - \beta_{R_0}(|x|)) A_{IJK}^{\alpha\beta} \phi^J \partial_\beta \Gamma^\mu u^K \right] \\ &\quad + \partial_\alpha \left[(1 - \beta_{R_0}(|x|)) B_{IJK}^{\alpha\beta\gamma} \partial_\gamma \phi^J \partial_\beta \Gamma^\mu u^K \right] + \partial_\alpha \left[(1 - \beta_{R_0}(|x|)) A_{IJK}^{\alpha\beta} \tilde{u}^J \partial_\beta \Gamma^\mu \psi_2^K \right] \\ &\quad + \partial_\alpha \left[(1 - \beta_{R_0}(|x|)) B_{IJK}^{\alpha\beta\gamma} \partial_\gamma \tilde{u}^J \partial_\beta \Gamma^\mu \psi_2^K \right]. \end{aligned}$$

We shall apply (2.5) to \tilde{w} , (2.7) to v , and (2.3), (2.4) to w . Upon doing so, it follows that the left side of (4.5) is controlled by

$$\begin{aligned} &\|[\Delta, \beta_{R_0}(|x|)]\Gamma^{\leq N} u\|_{L^2 L^2} + \|\beta'_{R_0} \phi \Gamma^{\leq N} \tilde{u}\|_{L^2 L^2} \\ &\quad + \|\beta'_{R_0} \partial^{\leq 1} \phi \Gamma^{\leq N} \partial u\|_{L^2 L^2} + \|\beta'_{R_0} \partial^{\leq 1} \tilde{u} \Gamma^{\leq N} \partial \psi_2\|_{L^2 L^2} \\ &+ \int_0^t \|\phi \Gamma^{\leq N} \tilde{u}\|_{L^2(\mathbb{R}^4 \setminus \mathcal{K})} ds + \int_0^t \|\partial^{\leq 1} \phi \Gamma^{\leq N} \partial u\|_{L^2(\mathbb{R}^4 \setminus \mathcal{K})} ds + \int_0^t \|\partial^{\leq 1} \tilde{u} \Gamma^{\leq N} \partial \psi_2\|_{L^2(\mathbb{R}^4 \setminus \mathcal{K})} ds \\ &\quad + \langle t \rangle^{-\delta} \|r^{-1} \Gamma^{\leq N/2} \phi \Gamma^{\leq N-1} \partial \tilde{u}\|_{L^1 L^1 L^2} + \|r^{-\frac{5}{4}} \Gamma^{\leq N/2} \phi \Gamma^{\leq N-1} \partial \tilde{u}\|_{L^1 L^1 L^2} \\ &\quad + \langle t \rangle^{-\delta} \|r^{-1} \Gamma^{\leq N/2} \partial \phi \Gamma^{\leq N} \partial^{\leq 1} \tilde{u}\|_{L^1 L^1 L^2} + \|r^{-\frac{5}{4}} \Gamma^{\leq N/2} \partial \phi \Gamma^{\leq N} \partial^{\leq 1} \tilde{u}\|_{L^1 L^1 L^2} \\ &\quad + \langle t \rangle^{-\delta} \|r^{-1} \Gamma^{\leq N/2} \partial \tilde{u} \Gamma^{\leq N} \partial^{\leq 1} \phi\|_{L^1 L^1 L^2} + \|r^{-\frac{5}{4}} \Gamma^{\leq N/2} \partial \tilde{u} \Gamma^{\leq N} \partial^{\leq 1} \phi\|_{L^1 L^1 L^2} \\ &+ \sum_{j=1,2} \left(\langle t \rangle^{-\delta} \|r^{-1} \Gamma^{\leq N/2} \partial^{\leq 1} \tilde{u} \Gamma^{\leq N} \partial \psi_j\|_{L^1 L^1 L^2} + \|r^{-\frac{5}{4}} \Gamma^{\leq N/2} \partial^{\leq 1} \tilde{u} \Gamma^{\leq N} \partial \psi_j\|_{L^1 L^1 L^2} \right) \\ &+ \sum_{j=1,2} \left(\langle t \rangle^{-\delta} \|r^{-1} \Gamma^{\leq N/2+1} \partial \psi_j \Gamma^{\leq N} \partial^{\leq 1} \tilde{u}\|_{L^1 L^1 L^2} + \|r^{-\frac{5}{4}} \Gamma^{\leq N/2+1} \partial \psi_j \Gamma^{\leq N} \partial^{\leq 1} \tilde{u}\|_{L^1 L^1 L^2} \right) \\ &\quad + \langle t \rangle^{-\delta} \|r^{-1} \Gamma^{\leq N/2} \phi \Gamma^{\leq N-1} \partial^2 u\|_{L^1 L^1 L^2} + \|r^{-\frac{5}{4}} \Gamma^{\leq N/2} \phi \Gamma^{\leq N-1} \partial^2 u\|_{L^1 L^1 L^2} \\ &\quad + \langle t \rangle^{-\delta} \|r^{-1} \Gamma^{\leq N/2} \partial \phi \Gamma^{\leq N} \partial u\|_{L^1 L^1 L^2} + \|r^{-\frac{5}{4}} \Gamma^{\leq N/2} \partial \phi \Gamma^{\leq N} \partial u\|_{L^1 L^1 L^2} \\ &\quad + \langle t \rangle^{-\delta} \|r^{-1} \Gamma^{\leq N/2+1} \partial u \Gamma^{\leq N} \partial^{\leq 1} \phi\|_{L^1 L^1 L^2} + \|r^{-\frac{5}{4}} \Gamma^{\leq N/2+1} \partial u \Gamma^{\leq N} \partial^{\leq 1} \phi\|_{L^1 L^1 L^2}. \quad (4.17) \end{aligned}$$

By (4.4), the first term is controlled by the right side of (4.5). By a Sobolev embedding, the second, third, and fourth term are controlled by

$$\begin{aligned} & \|\beta_{2R_0} \partial^{\leq 3} \partial \phi\|_{L^\infty L^2} \left(\|\langle x \rangle^{-3/4} \Gamma^{\leq N} \tilde{u}\|_{L^2 L^2} + \|\langle x \rangle^{-3/4} \Gamma^{\leq N} \partial u\|_{L^2 L^2} \right) \\ & \quad + \|\beta_{2R_0} \partial^{\leq 3} \partial \tilde{u}\|_{L^\infty L^2} \|\langle x \rangle^{-3/4} \Gamma^{\leq N} \partial \psi_2\|_{L^2 L^2} \\ & \lesssim M_3[\phi] \left(M_N[\tilde{u}] + M_N[u] \right) + M_3[\tilde{u}] M_N[\psi_2]. \end{aligned}$$

The fifth and sixth terms are bounded by applying (3.1) to the lower order factor and applying the Cauchy-Schwarz inequality. This gives that these terms are controlled by

$$\begin{aligned} & \|\langle x \rangle^{-3/4} \Gamma^{\leq 3} \partial^{\leq 1} \phi\|_{L^2 L^2} \left(\|\langle x \rangle^{-3/4} \Gamma^{\leq N} \tilde{u}\|_{L^2 L^2} + \|\langle x \rangle^{-3/4} \Gamma^{\leq N} \partial u\|_{L^2 L^2} \right) \\ & \lesssim M_3[\phi] \left(M_N[\tilde{u}] + M_N[u] \right). \end{aligned}$$

The same argument shows that the seventh term in (4.17) is $\mathcal{O}(M_3[\tilde{u}]M_N[\psi_2])$.

To control the remaining terms, except those that involve $\partial^2 u$, we shall show

$$\begin{aligned} \langle t \rangle^{-\delta} \|r^{-1} \partial^{\leq 1} \Phi \partial \Psi\|_{L^1 L^1 L^2} + \|r^{-\frac{5}{4}} \partial^{\leq 1} \Phi \partial \Psi\|_{L^1 L^1 L^2} & \lesssim M_2[\Phi](t) M_1[\Psi](t) \\ & \quad + M_2[\Phi](t) \|\langle x \rangle^{3/4} \square \Psi\|_{L^2 L^2} \quad (4.18) \end{aligned}$$

and

$$\begin{aligned} \langle t \rangle^{-\delta} \|r^{-1} \partial^{\leq 1} \Phi \partial \Psi\|_{L^1 L^1 L^2} + \|r^{-\frac{5}{4}} \partial^{\leq 1} \Phi \partial \Psi\|_{L^1 L^1 L^2} & \lesssim M_0[\Phi](t) M_3[\Psi](t) \\ & \quad + M_0[\Phi](t) \|\langle x \rangle^{3/4} \Gamma^{\leq 2} \square \Psi\|_{L^2 L^2}. \quad (4.19) \end{aligned}$$

Proof of (4.18) and (4.19). We divide the analysis into the region where $|x| < c_1 s/8$ and $|x| \geq c_1 s/8$. Applying Sobolev embeddings and the Schwarz inequality on \mathbb{S}^3 , these terms are bounded by

$$\begin{aligned} & \langle t \rangle^{-\delta} \|r^{-1} \partial^{\leq 1} \Phi \partial \Psi\|_{L^1 L^1 L^2} + \|r^{-\frac{5}{4}} \partial^{\leq 1} \Phi \partial \Psi\|_{L^1 L^1 L^2} \\ & \lesssim \sum_{j < \log t} \sum_{k < j - \tilde{C}} 2^{-k} \|\Gamma^{\leq 2} \partial^{\leq 1} \Phi\|_{L^2_{2^j} L^2_{2^k}} \|\partial \Psi\|_{L^2_{2^j} L^2_{2^k}} \\ & \quad + \langle t \rangle^{-\delta} \int_0^t \int_{r \geq c_1 s/4} \langle r \rangle^{-1} \|\Gamma^{\leq 2} \partial^{\leq 1} \Phi(t, r \cdot)\|_{L^2(\mathbb{S}^3)} \|\partial \Psi(t, r \cdot)\|_{L^2(\mathbb{S}^3)} r^2 dr ds \\ & \quad + \int_0^t \int_{r \geq c_1 s/4} \langle r \rangle^{-5/4} \|\Gamma^{\leq 2} \partial^{\leq 1} \Phi(t, r \cdot)\|_{L^2(\mathbb{S}^3)} \|\partial \Psi(t, r \cdot)\|_{L^2(\mathbb{S}^3)} r^2 dr ds \quad (4.20) \end{aligned}$$

for some fixed $\tilde{C} > 0$. Here we have applied the Sobolev embedding in the fashion that will yield (4.18). To prove (4.19), we instead apply the Sobolev estimate to Ψ and proceed with the same steps.

Applying the Schwarz inequality, the second and third terms in the right side of (4.20) are controlled by

$$\left(\langle t \rangle^{-\delta} \int_0^t \langle s \rangle^{-1+\delta} ds + \int_0^t \langle s \rangle^{-5/4+\delta} ds \right) M_2[\Phi](t) M_0[\Psi](t) \lesssim M_2[\Phi](t) M_0[\Psi](t).$$

To bound the first term in the right side of (4.20), we note that (3.2) (with $h \equiv 0$) gives

$$\|\partial \Psi\|_{L^2_{2j} L^2_{2k}} \lesssim 2^{-k} \|\Gamma^{\leq 1} \Psi\|_{L^2_{2j} L^2_{2k}} + 2^k \|\square \Psi\|_{L^2_{2j} L^2_{2k}}$$

when $k < j - \tilde{C}$. Thus,

$$\begin{aligned} & \sum_{j < \log t} \sum_{k < j - \tilde{C}} 2^{-k} \|\Gamma^{\leq 2} \partial^{\leq 1} \Phi\|_{L^2_{2j} L^2_{2k}} \|\partial \Psi\|_{L^2_{2j} L^2_{2k}} \\ & \lesssim \sum_j \sum_{k < j - \tilde{C}} \left(2^{-2k} \|\Gamma^{\leq 2} \partial^{\leq 1} \Phi\|_{L^2_{2j} L^2_{2k}} \|\Gamma^{\leq 1} \Psi\|_{L^2_{2j} L^2_{2k}} + \|\Gamma^{\leq 2} \partial^{\leq 1} \Phi\|_{L^2_{2j} L^2_{2k}} \|\square \Psi\|_{L^2_{2j} L^2_{2k}} \right) \\ & \lesssim M_2[\Phi](t) M_1[\Psi](t) + M_2[\Phi](t) \|\langle x \rangle^{3/4} \square \Psi\|_{L^2 L^2}, \end{aligned}$$

where we have applied the Cauchy-Schwarz inequality to sum over k, j . \square

Examining the remaining terms of (4.17), we see that (4.18) gives us

$$\begin{aligned} \langle t \rangle^{-\delta} \|r^{-1} \Gamma^{\leq N/2} \phi \Gamma^{\leq N-1} \partial \tilde{u}\|_{L^1 L^1 L^2} + \|r^{-5/4} \Gamma^{\leq N/2} \phi \Gamma^{\leq N-1} \partial \tilde{u}\|_{L^1 L^1 L^2} \\ \lesssim M_{N/2+2}[\phi] M_N[\tilde{u}] + M_{N/2+2}[\phi] \|\langle x \rangle^{3/4} \Gamma^{\leq N-1} \square \tilde{u}\|_{L^2 L^2} \end{aligned}$$

and

$$\begin{aligned} \langle t \rangle^{-\delta} \|r^{-1} \Gamma^{\leq N/2} \partial^{\leq 1} \tilde{u} \Gamma^{\leq N} \partial \psi_j\|_{L^1 L^1 L^2} + \|r^{-5/4} \Gamma^{\leq N/2} \partial^{\leq 1} \tilde{u} \Gamma^{\leq N} \partial \psi_j\|_{L^1 L^1 L^2} \\ \lesssim M_{N/2+2}[\tilde{u}] M_{N+1}[\psi_j] + M_{N/2+2}[\tilde{u}] \|\langle x \rangle^{3/4} \Gamma^{\leq N} \square \psi_j\|_{L^2 L^2}. \end{aligned}$$

And from (4.19), we obtain

$$\begin{aligned} \langle t \rangle^{-\delta} \|r^{-1} \Gamma^{\leq N/2} \partial \phi \Gamma^{\leq N} \partial^{\leq 1} \tilde{u}\|_{L^1 L^1 L^2} + \|r^{-5/4} \Gamma^{\leq N/2} \partial \phi \Gamma^{\leq N} \partial^{\leq 1} \tilde{u}\|_{L^1 L^1 L^2} \\ \lesssim M_N[\tilde{u}] M_{N/2+3}[\phi] + M_N[\tilde{u}] \|\langle x \rangle^{3/4} \Gamma^{\leq N/2+2} \square \phi\|_{L^2 L^2}, \end{aligned}$$

$$\begin{aligned} \langle t \rangle^{-\delta} \|r^{-1} \Gamma^{\leq N/2} \partial \tilde{u} \Gamma^{\leq N} \partial^{\leq 1} \phi\|_{L^1 L^1 L^2} + \|r^{-5/4} \Gamma^{\leq N/2} \partial \tilde{u} \Gamma^{\leq N} \partial^{\leq 1} \phi\|_{L^1 L^1 L^2} \\ \lesssim M_N[\phi] M_{N/2+3}[\tilde{u}] + M_N[\phi] \|\langle x \rangle^{3/4} \Gamma^{\leq N/2+2} \square \tilde{u}\|_{L^2 L^2}, \end{aligned}$$

$$\begin{aligned} \langle t \rangle^{-\delta} \|r^{-1} \Gamma^{\leq N/2+1} \partial \psi_j \Gamma^{\leq N} \partial^{\leq 1} \tilde{u}\|_{L^1 L^1 L^2} + \|r^{-5/4} \Gamma^{\leq N/2+1} \partial \psi_j \Gamma^{\leq N} \partial^{\leq 1} \tilde{u}\|_{L^1 L^1 L^2} \\ \lesssim M_N[\tilde{u}] M_{N/2+4}[\psi_j] + M_N[\tilde{u}] \|\langle x \rangle^{3/4} \Gamma^{\leq N/2+3} \square \psi_j\|_{L^2 L^2}, \end{aligned}$$

$$\begin{aligned} \langle t \rangle^{-\delta} \|r^{-1} \Gamma^{\leq N/2} \partial \phi \Gamma^{\leq N} \partial u\|_{L^1 L^1 L^2} + \|r^{-\frac{5}{4}} \Gamma^{\leq N/2} \partial \phi \Gamma^{\leq N} \partial u\|_{L^1 L^1 L^2} \\ \lesssim M_N[u] M_{N/2+3}[\phi] + M_N[u] \langle x \rangle^{3/4} \Gamma^{\leq N/2+2} \square \phi \|_{L^2 L^2}, \end{aligned}$$

and

$$\begin{aligned} \langle t \rangle^{-\delta} \|r^{-1} \Gamma^{\leq N/2+1} \partial u \Gamma^{\leq N} \partial^{\leq 1} \phi\|_{L^1 L^1 L^2} + \|r^{-\frac{5}{4}} \Gamma^{\leq N/2+1} \partial u \Gamma^{\leq N} \partial^{\leq 1} \phi\|_{L^1 L^1 L^2} \\ \lesssim M_N[\phi] M_{N/2+4}[u] + M_N[\phi] \langle x \rangle^{3/4} \Gamma^{\leq N/2+3} \square u \|_{L^2 L^2}. \end{aligned}$$

In order to control the remaining term

$$\langle t \rangle^{-\delta} \|r^{-1} \Gamma^{\leq N/2} \phi \Gamma^{\leq N-1} \partial^2 u\|_{L^1 L^1 L^2} + \|r^{-\frac{5}{4}} \Gamma^{\leq N/2} \phi \Gamma^{\leq N-1} \partial^2 u\|_{L^1 L^1 L^2}, \tag{4.21}$$

we shall argue as in (4.18), but it will be necessary to use a perturbation of \square in (3.2) so as to not exceed the allowable regularity. It is likely that estimates such as [14, Lemma 3.1] could be used as an alternative to (3.2) for these cases, but we do not explore such here.

We again consider $|x| < c_1 s/8$ and $|x| \geq c_1 s/8$ separately. Sobolev embeddings as above give that (4.21) is

$$\begin{aligned} \lesssim \sum_{j < \log t} \sum_{k < j - \tilde{C}} 2^{-k} \|\Gamma^{\leq N/2+2} \phi\|_{L^2_j L^2_{2^k}} \|\Gamma^{N-1} \partial^2 u\|_{L^2_j L^2_{2^k}} \\ + \langle t \rangle^{-\delta} \int_0^t \int_{r \geq c_1 s/4} \langle r \rangle^{-1} \|\Gamma^{\leq N/2+2} \phi(t, r \cdot)\|_{L^2(\mathbb{S}^3)} \|\Gamma^{\leq N} \partial u(t, r \cdot)\|_{L^2(\mathbb{S}^3)} r^2 dr ds \\ + \int_0^t \int_{r \geq c_1 s/4} \langle r \rangle^{-5/4} \|\Gamma^{\leq N/2+2} \phi(t, r \cdot)\|_{L^2(\mathbb{S}^3)} \|\Gamma^{\leq N} \partial u(t, r \cdot)\|_{L^2(\mathbb{S}^3)} r^2 dr ds \tag{4.22} \end{aligned}$$

with $\tilde{C} > 0$ as above. Using the Schwarz inequality and arguing as in the proof of (4.18), the last two terms are $\mathcal{O}(M_{N/2+2}[\phi](t) M_N[u](t))$.

It remains to bound the first term of (4.22). Here we again set $h^{IK, \alpha\beta} = -A_{IJK}^{\alpha\beta} \phi^J - B_{IJK}^{\gamma\alpha\beta} \partial_\gamma \phi^J$. Using (4.15) and (3.1), we have

$$\begin{aligned} 2^k \|\square_h \Gamma^{\leq N} u\|_{L^2_j L^2_{2^k}} \lesssim 2^{-k/2} \left(\|\Gamma^{\leq N/2+4} \phi\|_{L^2_j L^2_{2^k}} \|\Gamma^{\leq N} \tilde{u}\|_{L^\infty L^2} \right. \\ + \|\Gamma^{\leq N/2+3} \partial \tilde{u}\|_{L^\infty L^2} \|\Gamma^{\leq N} \partial^{\leq 1} \phi\|_{L^2_j L^2_{2^k}} + \|\Gamma^{\leq N/2+3} \partial \psi_1\|_{L^\infty L^2} \|\Gamma^{\leq N} \partial^{\leq 1} \tilde{u}\|_{L^2_j L^2_{2^k}} \\ + \|\Gamma^{\leq N/2+3} \partial^{\leq 1} \tilde{u}\|_{L^2_j L^2_{2^k}} \|\Gamma^{\leq N} \partial \psi_1\|_{L^\infty L^2} + \|\Gamma^{N/2+4} \phi\|_{L^2_j L^2_{2^k}} \|\Gamma^{\leq N} \partial u\|_{L^\infty L^2} \\ + \|\Gamma^{N/2+4} \partial u\|_{L^\infty L^2} \|\Gamma^{\leq N} \partial^{\leq 1} \phi\|_{L^2_j L^2_{2^k}} + \|\Gamma^{\leq N/2+3} \partial^{\leq 1} \tilde{u}\|_{L^2_j L^2_{2^k}} \|\Gamma^{\leq N+1} \partial \psi_2\|_{L^\infty L^2} \\ \left. + \|\Gamma^{\leq N/2+4} \partial \psi_2\|_{L^\infty L^2} \|\Gamma^{\leq N} \partial^{\leq 1} \tilde{u}\|_{L^2_j L^2_{2^k}} \right). \end{aligned}$$

By (3.2), for $k < j - \tilde{C}$, we have

$$\|\Gamma^{\leq N-1} \partial^2 u\|_{L_{2^j}^2 L_{2^k}^2} \lesssim 2^{-k} \|\Gamma^{\leq N} \partial u\|_{L_{2^j}^2 L_{2^k}^2} + 2^k \|\square_h \Gamma^{\leq N} u\|_{L_{2^j}^2 L_{2^k}^2}.$$

Using the previous two estimates, it thus follows that

$$\begin{aligned} \sum_{j < \log t} \sum_{k < j - \tilde{C}} 2^{-k} \|\Gamma^{\leq N/2+2} \phi\|_{L_{2^j}^2 L_{2^k}^2} \|\Gamma^{\leq N-1} \partial^2 u\|_{L_{2^j}^2 L_{2^k}^2} &\lesssim M_{N/2+2}[\phi] M_N[u] \\ &+ M_{N/2+2}[\phi] \left(\left[M_{N/2+4}[\phi] + M_{N/2+3}[\psi_1] + M_{N/2+4}[\psi_2] \right] M_N[\tilde{u}] \right. \\ &\quad \left. + M_{N/2+3}[\tilde{u}] \left[M_N[\phi] + M_N[\psi_1] + M_{N+1}[\psi_2] \right] \right. \\ &\quad \left. + M_{N/2+4}[\phi] M_N[u] + M_{N/2+4}[u] M_N[\phi] \right) \end{aligned}$$

by splitting the decay equally amongst the $L_{2^j}^2 L_{2^k}^2$ portions and using the Cauchy-Schwarz inequality to sum. This completes the proof. \square

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