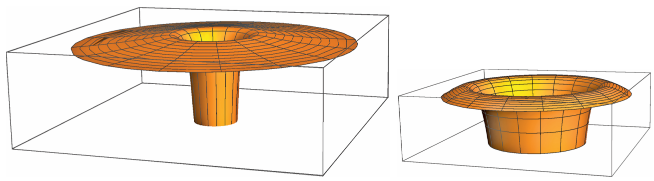
Local Energy Decay for the Wave Equation



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We want to understand how waves spread out in spaces that look like Minkowski space-time as you go to infinity. This is the Lorentzian analog of being asymptotically Euclidean. To begin, we consider solutions to the Minkowski wave equation on $\mathbb{R} \times \mathbb{R}^3$:

$$\Box u := \frac{\partial^2}{\partial t^2} u - \Delta u = 0$$

where Δ denotes the Laplacian

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}.$$

Such solutions enjoy a conserved energy $E(t) = E(0)$,

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^3} (\partial_t u)^2 + |\nabla_x u|^2 \, dx.$$

This energy is the same for all times; it does not decay.

On the other hand, if the domain of integration is restricted to a compact set, say $B_R = \{|x| \le R\}$, we expect the energy $E_{B_R}(t)$ within that set eventually to decrease in time. Imagine a stone thrown into the center of an infinite still ocean. After an initial splash, the waves move away, the center calms, and the energy within that set decreases. For some absolute constant C, if the total energy $E(0)$ is finite, say 1, then $E_{B_R}(t)$ has a finite integral with respect to t; indeed, its integral is bounded by CR. Such estimates originated in work of Cathleen Morawetz.

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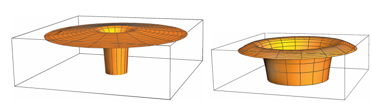
In joint works with Christopher Sogge and with Daniel Tataru, we proved that such estimates continue to hold for small perturbations of Minkowski space-time that decay as $|x| \to \infty$. Once such estimates are known, several other common measures of dispersion have been shown to follow in such spaces, including the so-called Strichartz estimates and pointwise decay estimates.

We now turn our attention to wave operators on such asymptotically flat backgrounds but where the size of the perturbation may not be universally small. Moreover, lower-order perturbations such as potentials are also allowed. Motivated by, e.g., Maxwell's equations, these lower-order perturbations are permitted to be complex valued.

We will describe two immediate obstructions to local energy decay. Recent work with Jacob Sterbenz and Tataru demonstrates that these are the only relevant obstructions. The first obstructions are certain possible eigenfunctions and resonances of the elliptic portion of the operator. The second obstruction is trapping, when a geodesic, and hence energy from a disturbance, can remain in a compact set for all time.

One place where trapping occurs is in black hole space-times from general relativity. In the simplest example, Schwarzschild space, as depicted in Figure 1, photons can orbit the black hole along the so-called photon sphere. Under magnification as in Figure 1(B), this appears as a ridge about the black hole. Recent work shows that because the trapped geodesics are unstable—nearby geodesics go off to infinity or into the black hole—local energy decay can be recovered with a small loss.



(A) Schwarzschild (B) Schwarzschild zoomed around the photon sphere

Figure 1. A depiction of the Schwarzschild space-time.

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This research is supported by the NSF CAREER award DMS-1054289.

DOI: http://dx.doi.org/10.1090/noti1437

Stable trapping, however, eliminates most local energy decay. Interesting examples of in-between scenarios have recently been discovered, and a more general theory remains to be examined.

Photo Credit

Photo of Jason Metcalfe, courtesy of Anita Hepditch.

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