Nonlinear Hyperbolic Equations in Infinite Homogeneous Waveguides

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May 25, 2007

- Quasilinear wave equations in $\mathbb{R}^n$:

$$\begin{cases} \Box w = (\partial_t^2 - \Delta) w = Q(w', w'') \\ (t, x) \in \mathbb{R}^{1+n} \\ w(0, \cdot\,) = f, \quad \partial_t w(0, \cdot\,) = g \end{cases}$$

  Q quadratic.

- Long time existence for small data:

  - $n \geq 4$ - Global existence

  - $n = 3$ - Almost global existence

- Main tools in the proof:

  - Energy estimate:

$$\|w'(t)\|_2 \lesssim \|w'(0)\|_2 + \int_0^t \|\Box w(s)\|_2 \, ds$$

  - Pointwise decay: $w \sim O\left(t^{-\frac{n-1}{2}}\right)$

- $\Omega \subset \mathbb{R}^d$ nonempty, bounded domain with smooth boundary

- Waveguide: $\mathbb{R}^n \times \Omega$

- d'Alembertian on $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^n \times \Omega$:

$$\Box = \partial_t^2 - (\Delta + \Delta_\Omega)$$

- Dirichlet IVBVP:

$$\begin{cases} \Box u = Q(u', u'') \\ u(t, x, y)|_{y \in \partial\Omega} = 0 \\ u(0, \cdot) = f, \quad \partial_t u(0, \cdot) = g \end{cases}$$

  - $u' = \nabla_{t,x,y} u$

  - Similar long time existence?

Klein-Gordon equation on $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$:

$$\begin{cases} (\Box + m^2)w = Q(w, w', w'') \\ w(0,\,\cdot\,) = f, \quad \partial_t w(0,\,\cdot\,) = g \end{cases}$$

- When $m > 0$, small data global existence is known for $n \geq 3$
  - Energy:

$$\|w'(t)\|_2 + m\|w(t)\|_2 \lesssim \|w'(0)\|_2 + m\|w(0)\|_2 + \int_0^t \|(\Box + m^2)w(s)\|_2 \, ds$$

  - Better pointwise decay: $w \sim O\left(t^{-\frac{n}{2}}\right)$

General Dirichlet IVBVP, $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^n \times \Omega$

$$\begin{cases} (\Box + m^2)u = Q(u, u', u'') \\ u(t, x, y)|_{y \in \partial\Omega} = 0 \\ u(0, \cdot) = f, \quad \partial_t u(0, \cdot) = g \end{cases}$$

Small initial data:

$$\text{supp} f, g \subset \{|x| \leq B\}$$

$$\|f\|_{H^N(\mathbb{R}^n \times \Omega)} + \|g\|_{H^{N-1}(\mathbb{R}^n \times \Omega)} \leq \varepsilon$$

Subject to standard compatibility conditions

<u>Theorem</u> [M.-Sogge-Stewart]: For $n \geq 3$ and $m \geq 0$, there is a smooth global solution provided $\varepsilon > 0$ is sufficiently small and N is sufficiently large.

- For $n \geq 5$, first proved by Lesky-Racke.
  - Relies on $L^p \to L^{p/(p-1)}$ decay estimates, similar to those of Marshall -Strauss - Wainger.
  - Existence proof similar to that of Shibata-Tsutsumi.

- Our proof:
  - Use linear estimates in boundaryless case which are independent of m
  - Project onto eigenspaces to prove linear estimates in the waveguide. Since there is no zero mode, these estimates will enjoy the $t^{-n/2}$ decay even if $m = 0$.
  - Show global existence using a continuity argument which couples these a priori estimates with energy estimates.

Commuting vector fields: $\{\Gamma\} = \{\Gamma_{t,x}\} \cup \{\partial_y\}$

$$\{\Gamma_{t,x}\} = \{\partial_t, \partial_x, \Omega_{jk} \,:\, 0 \le j < k \le n\}$$

$$\Omega_{jk} = x_j \partial_k - x_k \partial_j, \quad 1 \le j < k \le n$$

$$\Omega_{0k} = x_k \partial_t + t \partial_k, \quad 1 \le k \le n.$$

- $[\Box + m^2, \Gamma] = 0$

- $\Gamma_{t,x}$ preserve the boundary conditions; ∂_y doesn't necessarily preserve the b.c.

For simplicity, let's focus on the $n = 3$ argument.

<u>Boundaryless Linear Estimates</u> [Klainerman]: Let $\mu \geq 1$. Suppose $w(t,x) = 0$, $t \leq 2B$ and $(\Box_{\mathbb{R}^{1+3}} + \mu^2)w = 0$ for $|x| > t - B$. Then,

$$t^{3/2} \sup_x |w(t,x)| \lesssim \sum_{|\alpha| \leq 5} \sum_k \sup_{\tau \approx 2^k} 2^k \|\Gamma^\alpha_{t,x}(\Box + \mu^2) w(\tau, \cdot)\|_2.$$

- The implicit constant does not depend on μ!

- Use hyperbolic polar coordinates to reduce to the ODE $v'' + \mu^2 v = h$.

- Solving explicitly in $[a,b]$ shows that

$$\sup_{a \leq \rho \leq b} |v(\rho)| \leq |v(a)| + |v'(a)| + \frac{1}{\mu} \int_a^b |h(\rho)| \, d\rho$$

- The estimate follows from applying the above formula in dyadic time intervals.

Spectral Theory Review: Look at $-\Delta_\Omega$ with Dirichlet boundary conditions

- Since Ω is compact with smooth boundary, spectrum of $-\Delta_\Omega$ is discrete and non-negative. Due to the Dirichlet b.c., there is no zero mode. Denote the eigenvalues, $0 < \lambda_1^2 \leq \lambda_2^2 \leq \lambda_3^2 \leq \dots$

- Let $E_j : L^2(\Omega) \to L^2(\Omega)$ be the projection onto the jth eigenspace. I.e., for $h \in L^2(\Omega)$, $E_j h$ is smooth and

$$-\Delta_\Omega E_j h(x) = \lambda_j^2 E_j h(x).$$

- Plancherel:

$$\|h\|^2_{L^2(\Omega)} = \sum_{j=1}^{\infty} \|E_j h\|_2^2.$$

- Weyl formula: $\lambda_j \approx j^{1/d}$, $j = 2, 3, \dots$

Spectral Theory (CONT):

- For $h \in C^\infty$,

$$(1+j)^{2/d}\|E_j h\|_2 \lesssim \|(I-\Delta_\Omega)E_j h\|_2 = \|E_j(I-\Delta_\Omega)h\|_2$$

- $E_j(\Box + m^2)u(t,x,y) = (\partial_t^2 - \Delta + m^2 + \lambda_j^2)E_j u$

$$\sum_j \|(\partial_t^2 - \Delta + m^2 + \lambda_j^2)E_j u(t,x,\,\cdot\,)\|_2^2 = \|(\Box + m^2)u(t,x,\,\cdot\,)\|_2^2$$

<u>Elliptic regularity</u> ($N \geq 2$):

$$\sum_{|\alpha| \leq N} \|\partial_y^\alpha h\|_2 \lesssim \sum_{|\alpha| \leq N-2} \|\partial_y^\alpha \Delta_\Omega h\|_2 + \|h\|_2.$$

$$\begin{aligned}
&\sum_{|\alpha| \leq N} \|\partial_y^\alpha u(t, x, \cdot)\|_2 \\
&\qquad \lesssim \sum_{\substack{|\alpha|+|\beta| \leq N \\ |\beta| \leq 1}} \|\partial_{t,x}^\alpha \partial_y^\beta u(t, x, \cdot)\|_2 \\
&\qquad + \sum_{|\alpha| \leq N-2} \|\partial_{t,x,y}^\alpha (\Box + m^2) u(t, x, \cdot)\|_2
\end{aligned}$$

<u>Main Decay Estimate in Waveguide:</u> Suppose $u(t,x,y)=0$, $t \le 2B$ and $(\Box + m^2)u(t,x,y)=0$, $|x| > t-B$. Let $m \ge 0$.

$$
\begin{aligned}
& t^{3/2}|\Gamma^\beta u(t,x,y)| \\
& \lesssim \sum_{|\alpha| \le |\beta|+M} \sum_k \sup_{\tau \approx 2^k} 2^k \|\Gamma^\alpha(\Box + m^2)u(\tau,\,\cdot\,)\|_2 \\
& \quad + t^{3/2} \sum_{|\alpha| \le |\beta|+M-(9/2)} \|\Gamma^\alpha(\Box + m^2)u(t,\,\cdot\,)\|_2
\end{aligned}
$$

Here $M = 7 + (5/2)d$.

- WLOG, take $\Gamma^\beta = \partial_y^\beta$.

- Apply Sobolev in y to reduce to only L^2 theory for the Ω variables. Apply elliptic regularity to turn all ∂_y to $\partial_{t,x}$.

- Apply Plancherel and study each $E_j u$ individually.

- Use Weyl to see that

$$(1+j)^{2/d}\|E_j\partial_{t,x}^{\beta}u(t,x,\,\cdot\,)\|_{L^2(\Omega)}$$
$$\lesssim \sum_{|\alpha|\le 2+|\beta|}\|E_j\partial_{t,x}^{\alpha}u(t,x,\,\cdot\,)\|_{L^2(\Omega)}$$
$$+\sum_{|\alpha|\le|\beta|}\|E_j\partial_{t,x}^{\alpha}(\Box+m^2)u(t,x,\,\cdot\,)\|_{L^2(\Omega)}$$

- Recurse

$$\|E_j\partial_{t,x}^{\beta}u(t,s,\,\cdot\,)\|_{L^2(\Omega)}$$
$$\lesssim \frac{1}{(1+j)^2}\sum_{|\alpha|\le|\beta|+2d}\|E_j\partial_{t,x}^{\alpha}u(t,x,\,\cdot\,)\|_{L^2(\Omega)}$$
$$+\dots$$

This gains summability in j.

- Note that E_ju solves

$$(\partial_t^2-\Delta+m^2+\lambda_j^2)E_ju=E_j(\Box+m^2)u$$

and $m^2+\lambda_j^2>0$ for all j. So, we can apply the boundaryless estimate to each E_ju.

Energy estimates: For quasilinear equations, need energy estimates for

$$\Box_\gamma + m^2 = \Box + m^2 + \sum_{j,k=0}^{n+d} \gamma^{jk}(t,x,y)\partial_j\partial_k$$

where $|\gamma^{jk}|$ is small. For vanishing initial data,

$$\begin{aligned}&\|\partial_{t,x,y}u(t)\|_2 + m\|u(t)\| \\ &\leq 2\exp\left(\int_0^t 2\|\partial\gamma(s)\|_\infty ds\right)\int_0^t \|(\Box_\gamma + m^2)u(s,\,\cdot\,)\|_2 ds\end{aligned}$$

Higher order estimates follow trivially for $\Gamma = \Gamma_{t,x}$ and via elliptic regularity for ∂_y.

Global existence: Use local theory to reduce to vanishing initial data. Couple the above estimates using a standard continuity argument. To bound $\|u(t)\|_2$ when $m = 0$, simply use Poincaré's lemma: $\|u(t)\|_2 \lesssim \|\nabla_y u(t)\|_2$.

Neumann boundary conditions: $\partial_\nu u|_{y\in\partial\Omega} = 0$

- Must not have $n = 3$ global existence for the wave equation ($m = 0$). Even for simple nonlinearities $\Box u = Q(\nabla_x u)$. Let the data be independent of y. Then $u(t, x, y) = u(t, x)$. But the example of Sideris shows blow up in finite time for $Q(\nabla_x u) = |\nabla_x u|^2$.

- There is a zero mode! Results in weaker t decay for wave equations.

- Energy estimates in the quasilinear case have nontrivial nonvanishing boundary terms. Introduce an additional condition...

<u>Neumann nonlinear compatibility condition:</u> Expand

$$Q(u, u', u'') = A_l^{jk} \partial_l u \partial_j \partial_k u + A^{jk} u \partial_j \partial_k u + R(u, u').$$

Here $x_0 = t$, $x_{n+j} = y_j$.

Assume

$$A_l^{jk} \xi_l \eta_j \theta_k = 0, \quad A^{jk} \xi_j \theta_k = 0$$

if $(\theta, \xi, \eta) \in X$ with

$$X = \{\theta = (0, \nu), \xi \cdot \theta = 0, \eta \cdot \theta = 0, y \in \partial\Omega\}$$

This is a natural condition which guarantees that energy estimates hold.

<u>Theorem</u> [M.-Sogge-Stewart]: Let $n \geq 3$ and $m > 0$. Suppose further that the Neumann nonlinear compatibility condition holds. Then, there is a smooth global solution to the Neumann IVBVP provided $\varepsilon > 0$ is sufficiently small and N sufficiently large.

The same proof gives the pointwise decay for Klein-Gordon equations ($m > 0$). With the additional assumption, the energy estimates also hold.

For the wave equation $m = 0$, however, the proof of the pointwise bound does not follow. Notice that $\lambda_1^2 = 0$. Thus, $E_1 u$ solves an equation with "mass" $\mu = 0$, and thus, the boundaryless estimates do not apply.

Neumann wave equations: Need a different bound which we may apply to $E_1 u$ which solves

$$(\partial_t^2 - \Delta) E_1 u = E_1 \Box u.$$

New boundaryless estimate, under the same support conditions as before:

$$(1+t) \sup_x |\partial_{t,x} w(t,x)| \lesssim \sum_{|\alpha| \leq 2} \sum_k \sup_{\tau \approx 2^k} 2^k \|\Gamma_{t,x} (\partial_t^2 - \Delta) w(\tau, \cdot)\|_2$$

- Notice that we require a $\partial_{t,x}$ in the left.

- Consider separately the regions $|x| \geq t/2$ and $|x| \leq t/2$.

  - For the former, we apply some weighted Sobolev estimates due to Klainerman.

  - For the latter, we apply estimates in the spirit of Klainerman-Sideris.

Region $|x| \geq t/2$:

- Weighted Sobolev ($n = 3$):

$$\langle r \rangle |\partial \Gamma^{\alpha}_{t,x} w(t,x)| \lesssim \sum_{|\beta| \leq |\alpha|+2} \|\Gamma^{\beta} \partial w(t, \cdot)\|_2$$

  - Localize to annuli $|x| \approx R$.
  - Apply Sobolev on S^2. Compare volume elements on $\mathbb{R} \times S^2$ to $\mathbb{R}^3$.

- Apply the energy inequality for the wave equation to the right and divide the integral over dyadic regions.

Region $|x| \le t/2$:

- We first use a different weighted Sobolev estimate (Sideris) which gains a ∂:

$$\langle r\rangle^{1/2}|w(t,x)| \lesssim \sum_{|\alpha|\le 1} \|\Gamma^{\alpha}_{t,x}\partial w(t,\,\cdot\,)\|_2.$$

- Applying this to $\langle t-r\rangle|\partial\Gamma^{\alpha}w|$, we have

$$\begin{aligned}\langle r\rangle^{1/2}&\langle t-r\rangle|\partial\Gamma^{\alpha}_{t,x}w(t,x)| \\ &\lesssim \sum_{|\beta|\le|\alpha|+1} \|\Gamma^{\beta}_{t,x}w'(t,\,\cdot\,)\|_2 \\ &\quad + \sum_{|\beta|\le|\alpha|+1} \|\langle t-r\rangle\partial^2\Gamma^{\beta}w(t,\,\cdot\,)\|_2.\end{aligned}$$

- For the first term in the left, we may again apply the energy estimate.

- It remains only to estimate the latter term.

- To estimate the last term, use

$$\|\langle t-r\rangle \partial_{t,x}^2 \Gamma_{t,x}^{\alpha} w(t,\,\cdot\,)\|_2 \lesssim \sum_{|\beta|\le|\alpha|+1} \|\Gamma^{\beta} w'(t,\,\cdot\,)\|_2 + \sum_{|\beta|\le|\alpha|} \|\langle t+r\rangle \Box \Gamma^{\beta} w(t,\,\cdot\,)\|_2$$

  The main estimate follows by applying the energy inequality to the first term, and by summing the second term over dyadic intervals and using the assumed support properties of $\Box w$.

- Estimates of this type were first proved by Klainerman-Sideris in the context of multiple speed wave equations. There, the allowable vector fields is different. Instead of the hyperbolic rotations Ω_{0k} in the right, they allow for occurrences of the scaling vector field $L = t\partial_t + r\partial_r$.

- Using a straightforward integration by parts argument, the above follows from the differential inequality

$$\langle t-r\rangle|\Delta w(t,x)| \lesssim \sum_{|\alpha|\le 1} |\partial\Gamma^\alpha w(t,x)| + (t+r)|\Box w(t,x)|$$

as well as the corresponding ones with Δ replaced by $\partial_t\partial_{t,x}$. These were similarly shown by Klainerman-Sideris with the appropriate substitution of vector field.

<u>Main pointwise decay:</u> We can apply this new linear decay for $E_1 u$ which solves a wave equation, and use the previously established decay estimates for $E_j u$.

$$(1+t)|\Gamma^\beta\partial_{t,x}u| \lesssim \sum_{|\alpha|\le|\beta|+M}\sum_k \sup_{\tau\approx 2^k} 2^k\|\Gamma^\alpha\Box u(\tau,\,\cdot\,)\|_2 + \dots$$

with $M=(5/2)d+8$.

Neumann almost global existence: $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \Omega$

$$\begin{cases} \Box u = Q(\partial_{t,x} u, \partial^2_{t,x} u) \\ \partial_\nu u(t, x, y)|_{y \in \partial\Omega} = 0 \\ u(0, x, y) = f(x, y), \quad \partial_t u(0, x, y) = g(x, y) \end{cases}$$

Theorem [M.-Stewart]: For smooth, small (size $\sim \varepsilon$) data satisfying the usual compatibility conditions, there is a unique smooth solution u in $[0, T_\varepsilon)$ with $T_\varepsilon = \exp(\kappa/\varepsilon)$.

- We do not allow for dependence on $\partial_y u$ in the nonlinearity.

- As such, we do not need to require the Neumann nonlinear compatibility.

Useful, though naive, observation: Suppose $d = 1$. I.e. $\Omega = [a, b]$. Now we may easily allow for the dependence on ∂_y. I.e., for $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^3 \times [a, b]$, look at

$$\begin{cases} \Box u = Q(\partial u, \partial^2 u) \\ \partial_y u(t, x, a) = \partial_y u(t, x, b) = 0 \\ u(0, x, y) = f(x, y), \quad \partial_t u(0, x, y) = g(x, y) \end{cases}$$

Theorem [M.-Stewart]: Let the data be smooth, small, and satisfy the compatibility conditions. Suppose further that Q satisfies the Neumann nonlinear compatibility condition. Then, there is a unique smooth solution u in $[0, T_\varepsilon)$.

- The problem with allowing ∂_y before was that our decay estimate required $\partial_{t,x}$. Note, however, when $d = 1$, $\partial_y u$ solves a similar equation with Dirichlet boundary conditions. So, we have better decay estimates already available.

Recent related works:

- Stewart: Robin boundary conditions with a Robin nonlinear compatibility

- Fang-Zhong: Klein-Gordon in $n = 1$, Ω a Zoll manifold

- Lesky-Racke: Maxwell equations and elasticity.

Future extensions: $n \geq 4$ with Neumann conditions, wave equations with a null structure, allowing ∂_y more generally, ...