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LOCALIZED ENERGY ESTIMATES FOR WAVE EQUATIONS ON HIGH-DIMENSIONAL SCHWARZSCHILD SPACE-TIMES

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Abstract. The localized energy estimate for the wave equation is known to be a fairly robust measure of dispersion. Recent analogs on the $(1+3)$-dimensional Schwarzschild space-time have played a key role in a number of subsequent results, including a proof of Price's law. In this article, we explore similar localized energy estimates for wave equations on $(1+n)$-dimensional hyperspherical Schwarzschild space-times.

1. Introduction

One of the more robust measures of dispersion for the wave equation is the so-called localized energy estimates. These estimates have played a role in understanding scattering theory, as a means of summing local in time Strichartz estimates to obtain global in time Strichartz estimates on a compact set, as an essential tool for proving long-time existence of quasilinear wave equations in exterior domains, and as a means to handle errors in certain parametrix constructions.

Estimates of this form originated in [30], where it was shown that solutions to the constant coefficient, homogeneous wave equation

$$(\partial_t^2 - \Delta)u = 0, \quad u(0) = u_0, \quad \partial_t u(0) = u_1$$

on $\mathbb{R}_+ \times \mathbb{R}^n$, $n \geq 3$, satisfy

$$\int_0^T \int_{\mathbb{R}^n} \frac{1}{|x|} |\not\nabla u|^2(t,x)\, dx\, dt \lesssim \|\nabla u_0\|_2^2 + \|u_1\|_2^2,$$

where $\not\nabla$ denotes the normalized angular derivatives.[1] Though not the original method of proof, one can obtain this estimate by multiplying $\Box u$ by $\partial_r u + \frac{n-1}{2}\frac{u}{r}$, integrating over a space-time slab, and integrating by parts. One can also obtain control on ∂_r and ∂_t in a fixed dyadic annulus. Attempting to sum over these dyadic annuli comes at the cost of a logarithmic blow up in time. One may otherwise introduce an additional component of the weight that permits summability. In this case, it can be shown that for u as above and $n \geq 4$,

$$\|\langle x\rangle^{-1/2-} u'\|_{L^2_{t,x}([0,T]\times\mathbb{R}^n)} + \|\langle x\rangle^{-3/2} u\|_{L^2_{t,x}([0,T]\times\mathbb{R}^n)} \lesssim \|\nabla u_0\|_2 + \|u_1\|_2. \tag{1.1}$$

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[1] Here $\not\nabla = \frac{1}{r}\not\nabla_0$, where $\not\nabla_0$ are the derivatives on the unit sphere.

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Here $\langle x\rangle = \sqrt{1+|x|^2}$ denotes the Japanese bracket and $u' = (\partial_t u, \nabla_x u)$ represents the space-time gradient. An estimate akin to (1.1) also holds for $n = 3$, but in this case the weight $\langle x\rangle^{-3/2}$ in the second term on the left side must be replaced by $\langle x\rangle^{-3/2-}$. To prove (1.1), one multiplies the equation instead by $f(r)\partial_r u + \frac{n-1}{2}\frac{f(r)}{r}u$, where $f(r) = \frac{r}{r+2^j}$, $j \geq 0$, and integrates by parts. For fixed j, this yields an estimate for the left side over $|x| \approx 2^j$ with weight $\langle x\rangle^{-1/2}$ in the first term. Introducing the additional weight and summing over j produces (1.1). See, e.g., [20], [21], [22], [25], [32], [34], [36]. Analogous estimates have been shown for small, asymptotically flat, possibly time-dependent perturbations of the d'Alembertian in [1], [26], [27], [28] as well as for time-independent, asymptotically flat, nontrapping perturbations in e.g. [12], [13], [33].

A particular case of interest which does not fall into these latter categories is the wave equation on (1+3)-dimensional Schwarzschild space-times. While asymptotically flat and time independent, this metric is not, however, nontrapping.[2] There is trapping which occurs on the so-called photon sphere which necessitates a loss in the estimates as compared to those for the Minkowski wave equation. Despite this complication, a number of proofs of estimates akin to (1.1) exist for the wave equation on Schwarzschild space-times. See [5]-[10], [15], [16], and [24]. In [39], [17], and [3], these estimates have been extended to Kerr space-times with small angular momentums. Here an additional difficulty is encountered as [2] shows that the estimates will not follow from an analog of the argument above with any first order differential multiplier.

The goal of this article is to extend the known localized energy estimates on $(1+3)$-dimensional Schwarzschild space-times to $(1+n)$-dimensional Schwarzschild space-times for $n \geq 3$. There are multiple notions of the Schwarzschild space-time for $n \geq 4$, and we restrict our attention to the hyperspherical case. For a derivation and discussion of such black hole space times, see e.g. [37], [29]. An alternative notion of higher-dimensional Schwarzschild space-times is discussed e.g. in [19]. Here we shall not explore these hypercylindrical Schwarzschild manifolds or other notions of higher-dimensional black holes.

The exterior of a $(1+n)$-dimensional hyperspherical Schwarzschild black hole $(\mathcal{M}, g)$ is described by the manifold $\mathcal{M} = \mathbb{R} \times (r_s, \infty) \times \mathbb{S}^{d+2}$ and the line element

$$ds^2 = -\Big(1 - \Big(\frac{r_s}{r}\Big)^{d+1}\Big)dt^2 + \Big(1 - \Big(\frac{r_s}{r}\Big)^{d+1}\Big)^{-1} dr^2 + r^2 d\omega^2.$$

Here $d = n-3$,[3] r_s denotes the Schwarzschild radius (i.e. $r = r_s$ is the event horizon), and $d\omega$ is the surface measure on the sphere $\mathbb{S}^{d+2} = \mathbb{S}^{n-1}$. Here, the d'Alembertian is given by

$$\begin{aligned}\Box_g \phi &= \nabla^\alpha \partial_\alpha \phi \\ &= -\Big(1 - \Big(\frac{r_s}{r}\Big)^{d+1}\Big)^{-1} \partial_t^2 \phi + r^{-(d+2)} \partial_r \Big[r^{d+2} \Big(1 - \Big(\frac{r_s}{r}\Big)^{d+1}\Big) \partial_r \phi \Big] + \not\nabla \cdot \not\nabla \phi.\end{aligned}$$

[2]Here and throughout, the term "trapping" is used in the sense discussed in e.g. [35].

[3]We adopt the d notation to permit rapid comparison to the $(1+3)$-dimensional case $d = 0$.

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The Killing vector field ∂_t yields the conserved energy

$$E[\phi](t) = \int_{\mathbb{S}^{d+2}} \int_{r \geq r_s} \Big[\Big(1 - \Big(\frac{r_s}{r}\Big)^{d+1}\Big)^{-1} (\partial_t \phi)^2(t,r,\omega) + \Big(1 - \Big(\frac{r_s}{r}\Big)^{d+1}\Big)(\partial_r \phi)^2(t,r,\omega) + |\nabla\!\!\!/ \phi|^2(t,r,\omega)\Big] r^{d+2}\, dr\, d\omega.$$

That is, when $\Box_g \phi = 0$, $E[\phi](t) = E[\phi](0)$ for all t.

We now define our localized energy norm. To this end, we set

$$\|\phi\|_{LE}^2 = \int_0^\infty \int_{\mathbb{S}^{d+2}} \int_{r \geq r_s} \Big[c_r(r)\Big(1 - \Big(\frac{r_s}{r}\Big)^{d+1}\Big)(\partial_r \phi)^2(t,r,\omega) + c_\omega(r)|\nabla\!\!\!/ \phi|^2(t,r,\omega) + c_0(r)\phi^2(t,r,\omega)\Big] r^{d+2}\, dr\, d\omega\, dt, \tag{1.2}$$

where

$$c_r = \frac{1}{r^{d+3}\Big(1 - \log\Big(\frac{r-r_s}{r}\Big)\Big)^2}, \quad c_\omega = \frac{1}{r}\Big(\frac{r - r_{ps}}{r}\Big)^2,$$
$$c_0 = \Big(\frac{r - r_s}{r}\Big)^{-1} \frac{1}{r^3\Big(1 - \log\Big(\frac{r-r_s}{r}\Big)\Big)^4}.$$

Here $r_{ps} = \Big(\frac{d+3}{2}\Big)^{\frac{1}{d+1}} r_s$, which is the location of the photon sphere. The main result of this article then states that a localized energy estimate holds on $\mathcal{M}$.

Theorem 1.1. [4] *Let ϕ solve the homogeneous wave equation $\Box_g \phi = 0$ on a $(1+n)$-dimensional hyperspherical Schwarzschild manifold with $n \geq 4$. Then we have*

$$\sup_{t \geq 0} E[\phi](t) + \|\phi\|_{LE}^2 \lesssim E[\phi](0). \tag{1.3}$$

We note that by modifying the lower order portion of the multiplier which appears in the next section that it is straightforward to obtain an estimate on $\partial_t \phi$ as well. Moreover, one can obtain an analogous estimate for the inhomogeneous wave equation by placing the forcing term in an appropriate dual norm. The necessary decay of the coefficient c_r at ∞ can also be significantly improved. It is relatively simple to carry out these modifications, but for the sake of clarity, we omit the details.

As in the $(1 + 3)$-dimensional case, the higher-dimensional hyperspherical Schwarzschild space-times have a photon sphere where trapping occurs. Rays initially located on this sphere and moving initially tangent to it will stay on the surface for all times. Such trapping is an obstacle to many types of dispersive estimates, and the vanishing in the coefficient c_ω at the trapped set is a loss, when compared to the nontrapping Minkowski setting, which results from this trapping.

The proof follows by constructing an appropriate differential multiplier. This is carried out in the next section, and the construction most closely resembles that which appears in [24]. Some significant technicalities needed to be resolved, however, in order to find a construction which works in all dimensions $d \geq 1$. In particular, the most obvious modifications of the multiplier in the motivating work

[4] Upon completion of this work, the authors learned that a version of Theorem 1.1 was also obtained independently in the forthcoming [31].

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[24] do not yield expressions with the desired signs. Moreover, certain expressions which inspired the choice of multiplier in [24] in the $(1+3)$-dimensional case can no longer be integrated in closed form. To get around this, due to asymptotic flatness, a multiplier which is inspired by that for Minkowski space is chosen near infinity. Near the event horizon, the choice of multiplier is akin to that which occurs in [24]. It then remains to construct a C^2 transition between these choices which preserves the signs of a number of expressions that appear in the commutator.

The localized energy estimates on $(1+3)$-dimensional Schwarzschild manifolds have played a key role in a number of subsequent results. See e.g. [4], [5]-[10], [11], [14]-[18], [23], [24], [38], [39]. Theorem 1.1 permits possible generalizations of these studies to higher dimensions, and a portion of these will be explored in the first author's upcoming doctoral dissertation and other subsequent works.

2. Construction of the multiplier

Associated to $\Box_g$ is the energy-momentum tensor

$$Q_{\alpha\beta}[\phi] = \partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}\partial^\gamma\phi\partial_\gamma\phi$$

whose most important property is the divergence condition

$$\nabla^\alpha Q_{\alpha\beta}[\phi] = \partial_\beta\phi\Box_g\phi.$$

Contracting $Q_{\alpha\beta}$ with the radial vector field $X = f(r)\Big(1-\Big(\frac{r_s}{r}\Big)^{d+1}\Big)\partial_r$, we form the momentum density $P_\alpha[\phi,X] = Q_{\alpha\beta}[\phi]X^\beta$. Calculating the divergence of this quantity we have

$$\begin{aligned}\nabla^\alpha P_\alpha[\phi,X] = \Box_g\phi X\phi + f'(r)\Big(1-\Big(\frac{r_s}{r}\Big)^{d+1}\Big)^2(\partial_r\phi)^2 + \Big(\frac{r^{d+1}-\frac{d+3}{2}r_s^{d+1}}{r^{d+1}}\Big)\frac{f(r)}{r}|\nabla\!\!\!/\phi|^2 \\ - \frac{1}{2}\Big[\Big(1-\Big(\frac{r_s}{r}\Big)^{d+1}\Big)r^{-(d+2)}\partial_r(r^{d+2}f(r))\Big]\partial^\gamma\phi\partial_\gamma\phi.\end{aligned}$$

The last term involving the Lagrangian, $\partial^\gamma\phi\partial_\gamma\phi$, is not signed. In order to eliminate it, we utilize a lower order term in the multiplier. To this end, we modify the momentum density

$$\begin{aligned}\tilde{P}_\alpha[\phi,X] = P_\alpha[\phi,X] + \frac{1}{2}\Big[\Big(1-\Big(\frac{r_s}{r}\Big)^{d+1}\Big)r^{-(d+2)}\partial_r(r^{d+2}f(r))\Big]\phi\partial_\alpha\phi \\ - \frac{1}{4}\partial_\alpha\Big[\Big(1-\Big(\frac{r_s}{r}\Big)^{d+1}\Big)r^{-(d+2)}\partial_r(r^{d+2}f(r))\Big]\phi^2\end{aligned}$$

and recompute the divergence

$$\begin{aligned}\nabla^\alpha\tilde{P}_\alpha[\phi,X] = \Box_g\phi\Big[X\phi + \frac{1}{2}\Big\{\Big(1-\Big(\frac{r_s}{r}\Big)^{d+1}\Big)r^{-(d+2)}\partial_r(f(r)r^{d+2})\Big\}\phi\Big] \\ + \Big(1-\Big(\frac{r_s}{r}\Big)^{d+1}\Big)^2 f'(r)(\partial_r\phi)^2 + \Big(\frac{r^{d+1}-r_{ps}^{d+1}}{r^{d+1}}\Big)\frac{f(r)}{r}|\nabla\!\!\!/\phi|^2 \\ - \frac{1}{4}\nabla^\alpha\partial_\alpha\Big[\Big(1-\Big(\frac{r_s}{r}\Big)^{d+1}\Big)r^{-(d+2)}\partial_r(f(r)r^{d+2})\Big]\phi^2.\end{aligned} \tag{2.1}$$

Ideally one would choose $f(r)$ to be smooth, bounded, and so that the coefficients in the last three terms on the right are all nonnegative. Indeed, this is precisely what is done in e.g. [34] and [26] when $r_s = 0$, i.e. in Minkowski space-time. The localized energy estimate then follows from an application of the divergence

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theorem as well as an application of a Hardy inequality and the energy estimate in order to handle the time boundary terms. Unfortunately, in the current setting it does not appear possible to construct such an f. We shall instead construct f so that the last term on the right can be bounded below by a positive quantity minus a fractional multiple of the second term on the right.

We let $g(r) = \frac{r^{d+2}-r_{ps}^{d+2}}{r^{d+2}}$ and $h(r) = \ln\Big(\frac{r^{d+1}-r_s^{d+1}}{\frac{d+1}{2}r_s^{d+1}}\Big)$. The multiplier will be defined piecewise, and it will be convenient to parametrize in terms of the values of $h(r)$. To this end, we fix the notation r_θ to denote the value of r so that $h(r_\theta) = \theta$. Note, e.g., that $r_{-\infty} = r_s$ and $r_{ps} = r_0$. More explicitly, $r_\theta^{d+1} = r_s^{d+1}\Big(\frac{d+1}{2}e^\theta + 1\Big)$. An approximate multiplier is

$$g(r) + \frac{d+2}{d+3}\frac{r_{ps}r_s^{d+1}}{r^{d+2}}h(r).$$

We must, however, smooth out the logarithmic blow up at $r = r_s$. We must also smooth out $h(r)$ near ∞ in order to prevent a term in $f'(r)$ which has an unfavorable sign.

In order to accomplish this, set

$$a(x) = \begin{cases} -\frac{1}{\varepsilon}\frac{\varepsilon x+1}{\delta(\varepsilon x+1)-1} - \frac{1}{\varepsilon}, & x \le -\frac{1}{\varepsilon}, \\ x, & -\frac{1}{\varepsilon} \le x \le 0, \\ x - \frac{2}{3\alpha^2}x^3 + \frac{1}{5\alpha^4}x^5, & 0 \le x \le \alpha, \\ \frac{8\alpha}{15}, & x \ge \alpha. \end{cases}$$

Here $\alpha = 5 - \delta_0$ for some $0 < \delta_0 \ll 1$. Then, set

$$f(r) = g(r) + \frac{d+2}{d+3}\frac{r_{ps}r_s^{d+1}}{r^{d+2}}a(h(r)).$$

We notice that f is C^2 with the exception of a jump in the second derivative at $r_{-1/\varepsilon}$.

Using the assumption that $\Box_g\phi = 0$, integrating (2.1) over $[0,T]\times(r_s,\infty)\times\mathbb{S}^{d+2}$, and applying the divergence theorem, we have

(2.2)
$$\begin{aligned} &-\int\int f(r)\partial_t\phi\partial_r\phi\, r^{d+2}\,dr\,d\omega\Big|_0^T - \frac{1}{2}\int\int \frac{1}{r^{d+2}}\partial_r(f(r)r^{d+2})\phi\partial_t\phi\, r^{d+2}\,dr\,d\omega\Big|_0^T \\ &\quad = \int_0^T\int\int\Big\{\Big(1-\Big(\frac{r_s}{r}\Big)^{d+1}\Big)^2 f'(r)(\partial_r\phi)^2 + \Big(\frac{r^{d+1}-r_{ps}^{d+1}}{r^{d+1}}\Big)\frac{f(r)}{r}|\nabla\!\!\!/\phi|^2 \\ &\qquad\qquad + l(f)\phi^2\Big\}r^{d+2}\,dr\,d\omega\,dt \\ &\quad + \frac{1}{4}r_{-1/\varepsilon}^{d+2}\Big(1-\Big(\frac{r_s}{r_{-1/\varepsilon}}\Big)^{d+1}\Big)^2(f''(r_{-1/\varepsilon}^-) - f''(r_{-1/\varepsilon}^+))\int_0^T\int \phi^2|_{r=r_{-1/\varepsilon}}\,d\omega\,dt, \end{aligned}$$

where $l(f)(r)$ is defined (in the strong sense) on $(r_s,\infty)\backslash\{r_{-1/\varepsilon}, r_{ps}, r_\alpha\}$ via

$$l(f) = -\frac{1}{4}r^{-(d+2)}\partial_r\Big[\Big(1-\Big(\frac{r_s}{r}\Big)^{d+1}\Big)r^{d+2}\partial_r\Big\{\Big(1-\Big(\frac{r_s}{r}\Big)^{d+1}\Big)r^{-(d+2)}\partial_r(f(r)r^{d+2})\Big\}\Big].$$

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We first make a note about the boundary terms at $r_{-1/\varepsilon}$. Indeed, an elementary calculation shows that

$$\begin{aligned} f''(r^-_{-1/\varepsilon}) - f''(r^+_{-1/\varepsilon}) &= \frac{d+2}{d+3}\frac{r_{ps}r_s^{d+1}}{r^{d+2}_{-1/\varepsilon}} a''(-1/\varepsilon^-)(h'(r_{-1/\varepsilon}))^2 \\ &= 2\delta\varepsilon \frac{d+2}{d+3}\frac{r_{ps}r_s^{d+1}}{r^{d+2}_{-1/\varepsilon}}(h'(r_{-1/\varepsilon}))^2 \geq 0. \end{aligned}$$

We now proceed to show that the sum of the first three terms on the right of (2.2) produce a positive contribution. To do so, we will examine f on a case by case basis and show that

- $f'(r) > 0$ for all $r > r_s$.
- $f(r) < 0$ for $r < r_{ps}$ and $f(r) > 0$ for $r > r_{ps}$.
- $\int l(f)\phi^2\, r^{d+2}\, dr\, d\omega\, dt$ is bounded below by a positive term minus a fractional multiple of the $(\partial_r\phi)^2$ term and an $r_{-1/\varepsilon}$ boundary term.

By absorbing these latter pieces into those previously shown so as to positively contribute, the estimate shall nearly be at hand. It will only remain to examine the time boundary terms and, in particular, establish the Hardy inequality which will permit a direct application of conservation of energy.

Case 1. $r_s \leq r \leq r_{-1/\varepsilon}$. The multiplier in this region is constructed to smooth out the logarithmic blow up at the event horizon. This is the only case in which we shall not be able to just show that $l(f) \geq 0$.

Noting that $a(x) \leq -1/\varepsilon$ for $x \leq -1/\varepsilon$, we immediately see that $f(r) < 0$ on this range. We also compute

$$\begin{aligned} f'(r) = \frac{(d+2)r_{ps}^{d+2}}{r^{d+3}} - \frac{(d+2)^2}{d+3}\frac{r_{ps}r_s^{d+1}}{r^{d+3}}a(h(r)) \\ + \frac{d+2}{d+3}\frac{r_{ps}r_s^{d+1}}{r^{d+2}}\frac{1}{(\delta\varepsilon h(r)+\delta-1)^2}\frac{(d+1)r^d}{r^{d+1}-r_s^{d+1}}. \end{aligned} \tag{2.3}$$

Each of these summands is nonnegative on the given range, yielding the desired sign for $f'(r)$.

It remains to show an appropriate lower bound for the $l(f)\phi^2$ term of (2.2). Calculating $l(f)$ we find

$$\begin{aligned} l(g(r)) + l\Big(\frac{d+2}{d+3}\frac{r_{ps}r_s^{d+1}}{r^{d+2}}a(h(r))\Big) &= \frac{d+2}{4r^{2d+5}}\Big(dr^{2d+2} + (d+3)r_s^{d+1}r^{d+1} - (d+2)^2 r_s^{2d+2}\Big) \\ &+ \frac{(d+1)(d+2)}{2}\frac{r_{ps}r_s^{d+1}}{r^{2d+6}}(r_{ps}^{d+1} - r^{d+1})a'(h(r)) \\ &+ \frac{(d+1)^2(d+2)(d+5)}{4(d+3)}\frac{r_{ps}r_s^{d+1}}{r^{d+5}}a''(h(r)) \\ &- \frac{(d+1)^3(d+2)}{4(d+3)}\frac{r_{ps}r_s^{d+1}}{r^4}\frac{1}{r^{d+1}-r_s^{d+1}}a'''(h(r)). \end{aligned} \tag{2.4}$$

As $a'(h(r)) = (\delta\varepsilon h(r) + \delta - 1)^{-2}$ and $r < r_{ps}$ in this regime, the second term on the right has the desired sign. The third term on the right of (2.4) also has the desired sign as $a''(h(r)) = -2\delta\varepsilon/(\varepsilon\delta h(r) + \delta - 1)^3$ and $h(r) \leq -1/\varepsilon$ here.

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The key step is to control the contribution of the last term on the right of (2.4). We shall abbreviate $R(r) = \delta\varepsilon h(r) + \delta - 1$. By the Fundamental Theorem of Calculus, we observe that

$$\int_{r_s}^{r_{-1/\varepsilon}} \partial_r\Big(\frac{2\delta\varepsilon(d+1)^2(d+2)}{d+3}\frac{r_{ps}r_s^{d+1}}{r^2(R(r))^3}\phi^2\Big)\,dr = -\frac{2\delta\varepsilon(d+1)^2(d+2)}{d+3}\frac{r_{ps}r_s^{d+1}}{r_{-1/\varepsilon}^2}\phi(r_{-1/\varepsilon})^2.$$

Evaluating the derivative on the left side yields

$$\begin{aligned}\int_{r_s}^{r_{-1/\varepsilon}} &\frac{6\delta^2\varepsilon^2(d+1)^3(d+2)}{d+3}\frac{r_{ps}r_s^{d+1}}{r^2(R(r))^4}\frac{r^d}{r^{d+1}-r_s^{d+1}}\phi^2\,dr \\ &= -\int_{r_s}^{r_{-1/\varepsilon}} \frac{4\delta\varepsilon(d+1)^2(d+2)}{d+3}\frac{r_{ps}r_s^{d+1}}{r^3(R(r))^3}\phi^2\,dr \\ &\quad + \int_{r_s}^{r_{-1/\varepsilon}} \frac{4\delta\varepsilon(d+1)^2(d+2)}{d+3}\frac{r_{ps}r_s^{d+1}}{r^2(R(r))^3}\phi\partial_r\phi\,dr \\ &\qquad + \frac{2\delta\varepsilon(d+1)^2(d+2)}{d+3}\frac{r_{ps}r_s^{d+1}}{r_{-1/\varepsilon}^2}\phi(r_{-1/\varepsilon})^2.\end{aligned} \tag{2.5}$$

To the second term on the right, we apply the Schwarz inequality to obtain

$$\begin{aligned}\int_{r_s}^{r_{-1/\varepsilon}} &\frac{4\delta\varepsilon(d+1)^2(d+2)}{d+3}\frac{r_{ps}r_s^{d+1}}{r^2(R(r))^3}\phi\partial_r\phi\,dr \\ &\le \int_{r_s}^{r_{-1/\varepsilon}} \frac{3\delta^2\varepsilon^2(d+2)(d+1)^3}{d+3}\frac{r_{ps}r_s^{d+1}r^{d-2}}{(R(r))^4(r^{d+1}-r_s^{d+1})}\phi^2\,dr \\ &\quad + \frac{4}{3}\int_{r_s}^{r_{-1/\varepsilon}} \frac{(d+2)(d+1)}{d+3}\frac{r_{ps}r_s^{d+1}}{r^{d+2}}\frac{r^{d+1}-r_s^{d+1}}{(R(r))^2}(\partial_r\phi)^2\,dr.\end{aligned} \tag{2.6}$$

Plugging (2.6) into (2.5) yields

$$\begin{aligned}\int_{r_s}^{r_{-1/\varepsilon}} &\frac{3\delta^2\varepsilon^2(d+1)^3(d+2)}{d+3}\frac{r_{ps}r_s^{d+1}}{r^2(R(r))^4}\frac{r^d}{r^{d+1}-r_s^{d+1}}\phi^2\,dr \\ &\le -\int_{r_s}^{r_{-1/\varepsilon}} \frac{4\delta\varepsilon(d+1)^2(d+2)}{d+3}\frac{r_{ps}r_s^{d+1}}{r^3(R(r))^3}\phi^2\,dr \\ &\quad + \frac{4}{3}\int_{r_s}^{r_{-1/\varepsilon}} \frac{(d+2)(d+1)}{d+3}\frac{r_{ps}r_s^{d+1}}{r^{d+2}}\frac{r^{d+1}-r_s^{d+1}}{(R(r))^2}(\partial_r\phi)^2\,dr \\ &\qquad + \frac{2\delta\varepsilon(d+1)^2(d+2)}{d+3}\frac{r_{ps}r_s^{d+1}}{r_{-1/\varepsilon}^2}\phi(r_{-1/\varepsilon})^2.\end{aligned} \tag{2.7}$$

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This shows that

(2.8)

$$\Big(\frac{1}{4}+\frac{1}{48}\Big)\int_{\{r\in[r_s,r_{-1/\varepsilon}]\}}\frac{(d+1)^3(d+2)}{d+3}\frac{r_{ps}r_s^{d+1}}{r^4}\frac{1}{r^{d+1}-r_s^{d+1}}a'''(h(r))\phi^2\,r^{d+2}\,dr\,d\omega\,dt$$
$$\leq\frac{13}{12}\int_{\{r\in[r_s,r_{-1/\varepsilon}]\}}\frac{(d+1)^2(d+2)}{(d+3)}\frac{r_{ps}r_s^{d+1}}{r^{d+5}}a''(h(r))\phi^2\,r^{d+2}\,dr\,d\omega\,dt$$
$$+\frac{13}{18}\int_{\{r\in[r_s,r_{-1/\varepsilon}]\}}\frac{(d+2)(d+1)}{(d+3)}\frac{r_{ps}r_s^{d+1}}{r^{d+3}}\Big(1-\frac{r_s^{d+1}}{r^{d+1}}\Big)\frac{1}{(R(r))^2}(\partial_r\phi)^2\,r^{d+2}\,dr\,d\omega\,dt$$
$$+\frac{13}{12}\frac{\delta\varepsilon(d+1)^2(d+2)}{d+3}\frac{r_{ps}r_s^{d+1}}{r_{-1/\varepsilon}^2}\int\phi^2|_{r=r_{-1/\varepsilon}}\,d\sigma\,dt.$$

We now use (2.8) and the fact that $\frac{13}{12}\leq\frac{d+5}{4}$ for $d\geq 1$ to account for the last term in (2.4). We then see that

$$\int_{\{r\in[r_s,r_{-1/\varepsilon}]\}}l(f)\phi^2\,r^{d+2}\,dr\,d\omega\,dt\geq\int_{\{r\in[r_s,r_{-1/\varepsilon}]\}}l(g)\phi^2\,r^{d+2}\,dr\,d\omega\,dt \tag{2.9}$$
$$+\frac{1}{48}\int_{\{r\in[r_s,r_{-1/\varepsilon}]\}}\frac{(d+1)^3(d+2)}{d+3}\frac{r_{ps}r_s^{d+1}}{r^4}\frac{1}{r^{d+1}-r_s^{d+1}}a'''(h(r))\phi^2\,r^{d+2}\,dr\,d\omega\,dt$$
$$-\frac{13}{18}\int_{\{r\in[r_s,r_{-1/\varepsilon}]\}}\Big(1-\frac{r_s^{d+1}}{r^{d+1}}\Big)^2f'(r)(\partial_r\phi)^2\,r^{d+2}\,dr\,d\omega\,dt$$
$$-\frac{13}{12}\frac{\delta\varepsilon(d+1)^2(d+2)}{d+3}\frac{r_{ps}r_s^{d+1}}{r_{-1/\varepsilon}^2}\int\phi^2|_{r=r_{-1/\varepsilon}}\,d\sigma\,dt.$$

Here we have also used (2.3). As $l(g)$ remains bounded in the relevant region and as the coefficient in the integrand of the second term on the right of (2.9) is $\gtrsim\varepsilon^2e^{1/\varepsilon}\gg 1$ on the said region, the first term on the right can be controlled by a fraction of the second provided that ε is sufficiently small. This finally yields

$$\int_{\{r\in[r_s,r_{-1/\varepsilon}]\}}l(f)\phi^2\,r^{d+2}\,dr\,d\omega\,dt \tag{2.10}$$
$$\geq\frac{1}{96}\int_{\{r\in[r_s,r_{-1/\varepsilon}]\}}\frac{(d+1)^3(d+2)}{d+3}\frac{r_{ps}r_s^{d+1}}{r^4}\frac{1}{r^{d+1}-r_s^{d+1}}a'''(h(r))\phi^2\,r^{d+2}\,dr\,d\omega\,dt$$
$$-\frac{13}{18}\int_{\{r\in[r_s,r_{-1/\varepsilon}]\}}\Big(1-\frac{r_s^{d+1}}{r^{d+1}}\Big)^2f'(r)(\partial_r\phi)^2\,r^{d+2}\,dr\,d\omega\,dt$$
$$-\frac{13}{12}\frac{\delta\varepsilon(d+1)^2(d+2)}{d+3}\frac{r_{ps}r_s^{d+1}}{r_{-1/\varepsilon}^2}\int\phi^2|_{r=r_{-1/\varepsilon}}\,d\sigma\,dt.$$

The second term on the right can be bootstrapped into the positive contribution provided by the first term on the right of (2.2). The remaining boundary term at $r_{-1/\varepsilon}$ will be controlled at the end of this section using pieces from the subsequent case.

Case 2. $r_{-1/\varepsilon}\leq r\leq r_{ps}$. For r in this range, we simply have

$$f(r)=\frac{r^{d+2}-r_{ps}^{d+2}}{r^{d+2}}+\frac{d+2}{d+3}\frac{r_{ps}r_s^{d+1}}{r^{d+2}}\ln\Big(\frac{r^{d+1}-r_s^{d+1}}{\frac{d+1}{2}r_s^{d+1}}\Big),$$

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which is negative, as is desired in order to guarantee a positive contribution from the $|\nabla\!\!\!/\phi|^2$ term of (2.2). Moreover, we have

$$f'(r)=\frac{(d+2)r_{ps}^{d+2}}{r^{d+3}}+\frac{d+2}{d+3}\frac{r_{ps}r_s^{d+1}}{r^2}\frac{d+1}{r^{d+1}-r_s^{d+1}}-\frac{(d+2)^2}{d+3}\frac{r_s^{d+1}r_{ps}}{r^{d+3}}\ln\Big(\frac{r^{d+1}-r_s^{d+1}}{\frac{d+1}{2}r_s^{d+1}}\Big),$$

whose every term is positive for $r_s < r \le r_{ps}$.

It only remains to examine $l(f(r))$ for this region. Here, we first note that

$$l(g(r)) = \frac{d+2}{4r^{2d+5}}\Big(dr^{2d+2}+(d+3)r_s^{d+1}r^{d+1}-(d+2)^2r_s^{2d+2}\Big) \tag{2.11}$$

and

$$l\Big(\frac{d+2}{d+3}\frac{r_{ps}r_s^{d+1}}{r^{d+2}}h(r)\Big) = -\frac{(d+2)(d+1)}{4}\frac{r_{ps}r_s^{d+1}}{r^{2d+6}}\big(2r^{d+1}-(d+3)r_s^{d+1}\big). \tag{2.12}$$

Since (2.12) is nonnegative for $r \le r_{ps}$, we have

$$l\Big(\frac{d+2}{d+3}\frac{r_{ps}r_s^{d+1}}{r^{d+2}}h(r)\Big) \ge -\frac{(d+2)(d+1)}{4}\frac{r_s^{d+1}}{r^{2d+5}}\big(2r^{d+1}-(d+3)r_s^{d+1}\big)$$

for $r \le r_{ps}$. Summing this with (2.11), we have

$$l(f(r)) \ge \frac{d+2}{4r^{2d+5}}\big(r^{d+1}-r_s^{d+1}\big)\big(dr^{d+1}+r_s^{d+1}\big),$$

which is clearly nonnegative for $r \ge r_s$.

Case 3. $r_{ps} \le r \le r_\alpha$. This region corresponds precisely to $h(r) \in [0,\alpha]$. We, thus, see that

$$f(r) = \frac{r^{d+2}-r_{ps}^{d+2}}{r^{d+2}}+\frac{d+2}{d+3}\frac{r_{ps}r_s^{d+1}}{r^{d+2}}\frac{h(r)}{15\alpha^4}(3h(r)^4-10h(r)^2\alpha^2+15\alpha^4),$$

which is easily seen to be positive. Moreover,

$$f'(r)=\frac{(d+2)r_{ps}^{d+2}}{r^{d+3}}-\frac{(d+2)^2r_{ps}r_s^{d+1}}{(d+3)r^{d+3}}a(h(r))+\frac{d+2}{d+3}\frac{(d+1)r_{ps}r_s^{d+1}}{r^2(r^{d+1}-r_s^{d+1})}\frac{(h(r)^2-\alpha^2)^2}{\alpha^4}.$$

As $a(h(r))$ takes on a maximum value of $\frac{8\alpha}{15}$ and as $\frac{(d+2)(d+3)}{2}-\frac{(d+2)^2}{d+3}\frac{8\alpha}{15} > 0$ for $d \ge 1$ and $\alpha < 5$, the sum of the first two terms is positive. Also, as the last term is clearly positive, we see that $f'(r) > 0$ as desired.

It remains to verify that $l(f)$ is positive. We begin by calculating

$$\begin{aligned} l(f) = {} & \frac{d+2}{4r^{2d+5}}\Big(dr^{2d+2}+(d+3)r_s^{d+1}r^{d+1}-(d+2)^2r_s^{2d+2}\Big) \\ & + \frac{(d+2)(d+1)r_s^{d+1}r_{ps}}{4(d+3)r^{2d+6}(r^{d+1}-r_s^{d+1})}\Big(-2(d+3)(r^{d+1}-r_{ps}^{d+1})(r^{d+1}-r_s^{d+1})a'(h(r)) \\ & \qquad + (d+1)(d+5)r^{d+1}(r^{d+1}-r_s^{d+1})a''(h(r)) \\ & \qquad\qquad - (d+1)^2r^{2d+2}a'''(h(r))\Big). \end{aligned}$$

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Setting

$$\begin{aligned}
p(r) &= r(dr^{2d+2} + (d+3)r_s^{d+1}r^{d+1} - (d+2)^2 r_s^{2d+2}) \\
n_1(r) &= -r_{ps}r_s^{d+1}(d+1)(2r^{d+1} - r_s^{d+1}(d+3))\frac{(h(r)^2-\alpha^2)^2}{\alpha^4} \\
n_2(r) &= r_{ps}r_s^{d+1}\frac{(d+1)^2(d+5)}{d+3}r^{d+1}\frac{4h(r)(h(r)^2-\alpha^2)}{\alpha^4} \\
n_3(r) &= r_{ps}r_s^{d+1}\frac{(d+1)^3}{d+3}\frac{r^{2d+2}}{r^{d+1}-r_s^{d+1}}\cdot 4\frac{\alpha^2-3h(r)^2}{\alpha^4},
\end{aligned}$$

it remains to show that

$$p(r) + n_1(r) + n_2(r) + n_3(r) > 0.$$

The dominant term is $p(r)$, and we shall show that

$$\frac{1}{3}p(r) + n_1(r) > 0, \tag{2.13}$$

$$\frac{1}{2}p(r) + n_2(r) \geq 0, \tag{2.14}$$

$$\frac{1}{6}p(r) + n_3(r) \geq 0. \tag{2.15}$$

Proof of (2.13). Using the facts that we are in the regime $r \geq r_{ps}$ and that $(h(r)^2-\alpha^2)^2$ is maximized when $h(r)=0$, we have

$$\begin{aligned}
\frac{1}{3}p(r) + n_1(r) &\geq \frac{1}{3}r_{ps}(dr^{2d+2} + (d+3)r_s^{d+1}r^{d+1} - (d+2)^2 r_s^{2d+2}) \\
&\qquad - (d+1)r_{ps}r_s^{d+1}(2r^{d+1} - (d+3)r_s^{d+1}) \\
&= \frac{1}{3}r_{ps}\Big(dr^{2d+2} - (5d+3)r_s^{d+1}r^{d+1} + (2d^2+8d+5)r_s^{2d+2}\Big) \\
&= \frac{1}{3}r_{ps}\Big[d\Big(r^{d+1} - \frac{5d+3}{2d}r_s^{d+1}\Big)^2 + \frac{1}{4d}(d+1)^2(8d-9)r_s^{2d+2}\Big].
\end{aligned}$$

The last quantity is clearly positive, as desired, for $d > 1$.

For the case $d = 1$,

$$\begin{aligned}
\frac{1}{3}p(r) + n_1(r) &\geq \frac{1}{3}r(r^4 + 4r_s^2r^2 - 9r_s^4) - 2\sqrt{2}r_s^3(2r^2 - 4r_s^2) \\
&= \frac{1}{3}\Big(3\sqrt{2}r_s^5 - 13r_s^4(r-\sqrt{2}r_s) + 20\sqrt{2}r_s^3(r-\sqrt{2}r_s)^2 \\
&\qquad + 24r_s^2(r-\sqrt{2}r_s)^3 + 5\sqrt{2}r_s(r-\sqrt{2}r_s)^4 + (r-\sqrt{2}r_s)^5\Big) \\
&\geq \frac{1}{3}\Big(3\sqrt{2}r_s^5 - 13r_s^4(r-\sqrt{2}r_s) + 20\sqrt{2}r_s^3(r-\sqrt{2}r_s)^2\Big),
\end{aligned}$$

which is an everywhere positive quadratic. □

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Proof of (2.14). Here, again, we use the fact that we are studying the regime that $r \geq r_{ps}$ and that $h^2 - \alpha^2$ is minimized when $h(r) = 0$. It is thus obtained that

(2.16)
$$\begin{aligned}\frac{1}{2}p(r) + n_2(r) &\geq \frac{1}{2}r_{ps}(dr^{2d+2} + (d+3)r_s^{d+1}r^{d+1} - (d+2)^2 r_s^{2d+2}) \\ &\qquad - \frac{4r_{ps}r_s^{d+1}}{\alpha^2}\frac{(d+1)^2(d+5)}{d+3}r^{d+1}h(r) \\ &= \frac{1}{2}r_{ps}\Big(d(r^{d+1} - r_s^{d+1})^2 + 3(d+1)r_s^{d+1}(r^{d+1} - r_s^{d+1}) - (d+1)^2 r_s^{2d+2} \\ &\qquad - \frac{8r_s^{d+1}}{\alpha^2}\frac{(d+1)^2(d+5)}{d+3}r^{d+1}h(r)\Big).\end{aligned}$$

Here, we make the change of variables $x = h(r)$. Thus, $r^{d+1} - r_s^{d+1} = \frac{d+1}{2}r_s^{d+1}e^x$. The right side of (2.16) can be rewritten as

$$\frac{r_{ps}}{2}r_s^{2d+2}(d+1)^2\Big[\frac{d}{4}e^{2x} + \frac{3}{2}e^x - 1 - \frac{4}{\alpha^2}\frac{(d+5)(d+1)}{d+3}xe^x - \frac{8}{\alpha^2}\frac{d+5}{d+3}x\Big].$$

Setting

$$q(x) = \frac{d}{4}e^{2x} + \frac{3}{2}e^x - 1 - \frac{4}{\alpha^2}\frac{(d+5)(d+1)}{d+3}xe^x - \frac{8}{\alpha^2}\frac{d+5}{d+3}x$$

and noticing that $q(0) > 0$, it will suffice to show that $q'(x) \geq 0$ for x between 0 and α. We compute that

$$q'(x) = \frac{1}{2}e^x(3 + de^x) - \frac{4}{\alpha^2}\frac{d+5}{d+3}\Big[2 + (1+d)e^x(1+x)\Big].$$

As $\frac{d+5}{d+3} \leq \frac{3}{2}$ for $d \geq 1$ and as $1 + x \leq e^x$, it follows that

$$q'(x) \geq \frac{1}{2}e^x(3 + de^x) - \frac{6}{25}\Big(2e^x + (d+1)e^{2x}\Big) > 0, \quad d \geq 1,\ \alpha = 5.$$

The latter inequality follows from the fact that $\frac{d}{2} \geq \frac{6}{25}(d+1)$ when $d \geq 1$. Since the above inequality holds for $\alpha = 5$, by continuity, we have that it also holds for $\alpha = 5 - \delta_0$ for some $\delta_0 > 0$, which completes the proof. □

Proof of (2.15). Using the fact that we are examining the region $r \geq r_{ps}$ and rewriting $l(g)$ as in the previous case, we have

$$\begin{aligned}\frac{1}{6}p(r) + n_3(r) \geq \frac{1}{6}r_{ps}\Big[&d(r^{d+1} - r_s^{d+1})^2 + 3(d+1)r_s^{d+1}(r^{d+1} - r_s^{d+1}) - (d+1)^2 r_s^{2d+2} \\ &+ \frac{24r_s^{d+1}}{\alpha^2}\frac{(d+1)^3}{d+3}\Big(1 - \frac{3}{\alpha^2}h(r)^2\Big)\Big((r^{d+1} - r_s^{d+1}) + 2r_s^{d+1} + \frac{r_s^{2d+2}}{r^{d+1} - r_s^{d+1}}\Big)\Big].\end{aligned}$$

Using the fact that

$$\begin{aligned}\frac{24r_s^{d+1}}{\alpha^2}\frac{(d+1)^3}{d+3}\Big(1 - \frac{3}{\alpha^2}(h(r))^2\Big)\frac{r_s^{2d+2}}{r^{d+1} - r_s^{d+1}} &\geq -\frac{72r_s^{d+1}}{\alpha^4}\frac{(d+1)^3}{d+3}(h(r))^2\frac{r_s^{2d+2}}{r^{d+1} - r_s^{d+1}} \\ &\geq -\frac{144}{\alpha^4}r_s^{2d+2}\frac{(d+1)^2}{d+3}(h(r))^2\end{aligned}$$

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when $r \geq r_{ps}$, we obtain

$$\frac{1}{6}p(r)+n_3(r) \geq \frac{1}{6}r_{ps}\Big[d(r^{d+1}-r_s^{d+1})^2+3(d+1)r_s^{d+1}(r^{d+1}-r_s^{d+1})-(d+1)^2r_s^{2d+2} + \frac{24r_s^{d+1}}{\alpha^2}\frac{(d+1)^3}{d+3}\Big(1-\frac{3}{\alpha^2}h(r)^2\Big)\Big((r^{d+1}-r_s^{d+1})+2r_s^{d+1}\Big) - \frac{144}{\alpha^4}r_s^{2d+2}\frac{(d+1)^2}{d+3}(h(r))^2\Big].$$

Proceeding as above with the change of variables $x = h(r)$, this is

$$=\frac{(d+1)^2}{6}r_{ps}r_s^{2d+2}\Big[\frac{d}{4}e^{2x}+\frac{3}{2}e^x-1 + \frac{24}{\alpha^2}\frac{d+1}{d+3}\Big(1-\frac{3}{\alpha^2}x^2\Big)\Big(\frac{d+1}{2}e^x+2\Big)-\frac{144}{\alpha^4}\frac{1}{d+3}x^2\Big].$$

Setting

$$s(x)=\frac{d}{4}e^{2x}+\frac{3}{2}e^x-1+\frac{24}{\alpha^2}\frac{d+1}{d+3}\Big(1-\frac{3}{\alpha^2}x^2\Big)\Big(\frac{d+1}{2}e^x+2\Big)-\frac{144}{\alpha^4}\frac{1}{d+3}x^2,$$

we first note that $s(0) > 0$. For $x \leq 5$, we furthermore have

$$\begin{aligned} s'(x) &= \frac{1}{2\alpha^4(d+3)}\Big[24\alpha^2(d+1)^2e^x+\alpha^4(d+3)e^x(3+de^x) \\ &\qquad - 72x(8(d+2)+(d+1)^2e^x(x+2))\Big] \\ &\geq \frac{1}{2\alpha^4(d+3)}\Big[24\alpha^2(d+1)^2e^x+\alpha^4(d+3)e^x(3+de^x) \\ &\qquad - 72e^x(8(d+2)+7(d+1)^2e^x)\Big]. \end{aligned}$$

For $\alpha = 5$, this is

$$\frac{1}{1250(d+3)}e^x\Big(5073-504e^x+d(2499+867e^x)+d^2(600+121e^x)\Big),$$

which is easily seen to be positive for $d \geq 1$. By continuity, positivity also follows for $\alpha = 5-\delta_0$ provided that δ_0 is sufficiently small.

Case 4. $r \geq r_\alpha$. In this regime,

$$f(r)=\frac{r^{d+2}-r_{ps}^{d+2}}{r^{d+2}}+\frac{8\alpha}{15}\frac{d+2}{d+3}\frac{r_{ps}r_s^{d+1}}{r^{d+2}},$$

which is clearly positive. Moreover,

$$f'(r)=\frac{(d+2)r_s^{d+1}r_{ps}}{r^{d+3}}\Big(\frac{d+3}{2}-\frac{8\alpha}{15}\frac{d+2}{d+3}\Big),$$

which is also positive since $\alpha < 5 \leq \frac{15}{16}\frac{(d+3)^2}{d+2}$ for $d \geq 1$. Finally, we notice that $l(f(r)) = l(g(r))$ when in this case. Thus, as in the proof of (2.14), we have

$$l(f(r))=r_{ps}\Big(d(r^{d+1}-r_s^{d+1})^2+3(d+1)r_s^{d+1}(r^{d+1}-r_s^{d+1})-(d+1)^2r_s^{2d+2}\Big).$$

As $r^{d+1}-r_s^{d+1} \geq \frac{d+1}{2}r_s^{d+1}$ for $r \geq r_{ps}$, we see that $l(f(r)) > 0$ as desired.

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Boundary term at $r_{-1/\varepsilon}$. In order to finish showing that the right side of (2.2) is nonnegative, it remains to examine the $r_{-1/\varepsilon}$ boundary term in (2.10). Here, we simply utilize the Fundamental Theorem of Calculus to control these terms via the positive contributions of the first and third term on the right of (2.2) in the range $[r_{-1/\varepsilon}, r_{ps}]$. The scaling parameter δ insures the necessary smallness.

Fix a smooth cutoff β which is the identity for, say, $r \le r_{-1}$ and which vanishes for $r \ge r_{ps}$. Then, for $r \le r_{-1}$, we have

$$\phi(r) = -\int_r^{r_{ps}} \partial_s(\beta\phi)\, ds.$$

Using the Schwarz inequality, this yields

$$\phi^2(r) \lesssim \int_r^{r_{ps}} |\beta'|\phi^2\, ds - h(r)\int_r^{r_{ps}} (s^{d+1} - r_s^{d+1})\beta(\partial_r\phi)^2\, ds.$$

Applying this at $r_{-1/\varepsilon}$ yields

$$\varepsilon\phi^2(r_{-1/\varepsilon}) \lesssim \int_{r_{-1/\varepsilon}}^{r_{ps}} \nabla^\alpha \tilde{P}_\alpha[\phi, X] r^{d+2}\, dr.$$

Multiplying both sides by δ and integrating over $[0,T]\times\mathbb{S}^{d+2}$, we see that these boundary terms can be bootstrapped into the contributions of Case 2.

3. A Hardy inequality and the time boundary terms

In the previous section, we constructed a multiplier so that the right side of (2.2) provides a positive contribution. By inspection, the coefficients are easily seen to correspond to those in (1.2). What remains is to control the left side of (2.2) in terms of the initial energy. For the first term, this is straightforward. For the second term on the left side of (2.2), a Hardy-type inequality is employed, which shall be proved below.

For the first term in (2.2), we need only apply the Schwarz inequality to see that

$$\int f(r)\partial_t\phi(t,\,\cdot\,)\partial_r\phi(t,\,\cdot\,)\, r^{d+2}\, dr\, d\omega \lesssim E[\phi](t).$$

Thus, by conservation of energy, these terms are controlled by $E[\phi](0)$ as desired.

For the second term in (2.2), we again apply the Schwarz inequality. It remains to show that

$$\int \Big[\frac{1}{r^{d+2}}\partial_r(f(r)r^{d+2})\Big]^2 \Big(1 - \frac{r_s^{d+1}}{r^{d+1}}\Big)\phi^2(t,\,\cdot\,)\, r^{d+2}\, dr\, d\omega \lesssim E[\phi](t) \tag{3.1}$$

as a subsequent application of conservation of energy will complete the proof.

In order to show (3.1), we shall prove a Hardy-type inequality which is in the spirit of that which appears in [15]. Indeed, we notice that the coefficient in the integrand on the left side of (3.1) is $O((\log(r-r_s))^{-2}(r-r_s)^{-1})$ as $r \to r_s$ and is $O(r^{-2})$ as $r\to\infty$. Thus, it will suffice to show that

$$\int_{r_s}^\infty \frac{1}{r^2}\frac{1}{\Big(1-\log\Big(\frac{r-r_s}{r}\Big)\Big)^2\Big(\frac{r-r_s}{r}\Big)}\phi^2\, r^{d+2}\, dr \lesssim \int_{r_s}^\infty \Big(\frac{r-r_s}{r}\Big)(\partial_r\phi)^2\, r^{d+2}\, dr. \tag{3.2}$$

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To this end, we set

$$\rho(r) = \int_{r_s}^{r} \frac{x^d}{\left(1 - \log\left(\frac{x-r_s}{x}\right)\right)^2 \left(\frac{x-r_s}{x}\right)} \, dx.$$

Notice that $\rho(r) \sim r^{d+1}$ as $r \to \infty$ and $\rho(r) \sim \left[1 - \log\left(\frac{r-r_s}{r}\right)\right]^{-1}$ as $r \to r_s$.

Writing the left side of (3.2) as $\int \rho'(r)\phi^2 \, dr$, integrating by parts, and applying the Schwarz inequality, we have

$$\begin{aligned} \int \rho'(r)\phi^2 \, dr &= -2 \int \rho(r)\phi\partial_r\phi \, dr \\ &\lesssim \Big(\int \frac{(\rho(r))^2}{\rho'(r)} (\partial_r\phi)^2 \, dr \Big)^{1/2} \Big(\int \rho'(r)\phi^2 \, dr \Big)^{1/2}. \end{aligned}$$

This completes the proof of (3.2) as $\frac{(\rho(r))^2}{\rho'(r)} \sim r^{d+2}$ as $r \to \infty$ and $\frac{(\rho(r))^2}{\rho'(r)} \sim (r - r_s)$ as $r \to r_s$.

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