A Multiplier Approach to KSS Estimates and Applications

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KSS Estimate

[Keel-Smith-Sogge '00]

$$(\log(2+T))^{-1/2}\left\|\langle x\rangle^{-1/2}u'\right\|_{L_t^2L_x^2([0,T]\times\mathbb{R}^n)}+\left\|\langle x\rangle^{-1/2-}u'\right\|_{L_t^2L_x^2([0,T]\times\mathbb{R}^n)}$$
$$\le C\left(\|u'(0,\cdot)\|_{L^2(\mathbb{R}^n)}+\int_0^T\|\Box u(t,\cdot)\|_2\,dt\right)$$

- First used to show almost global existence for some semilinear wave equations in exterior domains

Example: Long Time Existence

$$\begin{aligned}&\Box u = Q(u'), \qquad (t,x)\in\mathbb{R}\times\mathbb{R}^3\\ &u(0,x)=f(x), \quad \partial_t u(0,x)=g(x)\end{aligned} \qquad (1)$$

Q is quadratic in its arguments. $u'=\nabla_{t,x}u$

Assume that $\sum_{|\alpha|\le N}\|Z^\alpha f\|_2+\sum_{|\alpha|\le N-1}\|Z^\alpha g\|_2\le\varepsilon$

$$Z=\{\partial_k,\Omega_{ij}=x_i\partial_j-x_j\partial_i\},\quad k=0,\ldots,3;\quad 1\le i<j\le 3$$

Example: Long Time Existence

- [John-Klainerman '84, Keel-Smith-Sogge '00] There is a unique almost global solution $u\in C^\infty([0,T_*)\times\mathbb{R}^3)$ where $T_*=\exp(c/\varepsilon)$
- Main decay estimate [Klainerman '86]:

$$\|h\|_{L^\infty(R/2\le|x|\le R)}\le CR^{-1}\sum_{|\alpha|\le 2}\|Z^\alpha h\|_{L^2(R/4\le|x|\le 2R)}$$

Sketch of proof

- Set up iteration $u_0=0,\quad \Box u_k=Q(u_{k-1}')$
- Let

$$M_k(T)=\sup_t\sum_{|\alpha|\le 10}\|Z^\alpha u_k'(t,\cdot)\|_2+(\log(2+T))^{-1/2}\sum_{|\alpha|\le 10}\left\|\langle x\rangle^{-1/2}Z^\alpha u_k'\right\|_{L_t^2L_x^2([0,T]\times\mathbb{R}^3)}$$

- Show inductively that this is bounded.
- There is a similar second step to show convergence (no details given).

Sketch of proof

- Use an energy estimate and the KSS estimate to bound $M_k(T_*)$ by

$$C\varepsilon+C\sum_{|\alpha|\le 10}\int_0^{T_*}\|Z^\alpha Q(u_{k-1}')(s,\cdot)\|_2\,ds$$

- Decomposing dyadically and applying the weighted Sobolev estimate to the lower order term, this is

$$\le C\varepsilon+C\sum_{|\alpha|\le 10}\left\|\langle x\rangle^{-1/2}Z^\alpha u_{k-1}'\right\|^2_{L_t^2L_x^2([0,T_*]\times\mathbb{R}^3)}\le C\varepsilon+C(\log(2+T_*))M_{k-1}^2$$

which (using induction) is bounded for the given lifespan.

Remarks on the proof

- A similar estimate can be used to show almost global existence ($n=3$) [KSS ’00] and global existence ($n\geq 4$) [M. ’04] in the exterior of a nontrapping obstacle.
- Key observation: The proof does not rely on Lorentz invariance. I.e., it does not require the vector fields: $\Omega_{0k}=x_k\partial_t+t\partial_k$

Proofs of the KSS Estimate

- Both of the previous proofs use scaling to reduce to showing a localized version:

$$\|u'\|_{L^2_tL^2_x([0,T]\times\{|x|\leq 1\})}\leq C\left(\|u'(0,\cdot)\|_{L^2(\mathbb{R}^n)}+\int_0^T\|\Box u(t,\cdot)\|_2\,dt\right)$$

  - In odd dimensions, this follows by decomposing time into unit slices and using Huygens’ principle (the backward light cones have finite overlap). [KSS ’00].
  - In all dimensions, one can use Plancherel [Smith-Sogge ’00].

Remarks on these proofs

- Disadvantages of these methods:
  - It is not clear how to allow variable coefficient wave equations. This complicates the existence proofs in the quasilinear case. Now one needs to use the scaling vector field $L=t\partial_t+r\partial_r$ and tools such as $L^\infty-L^1$ Hörmander type estimates. [Keel-Smith-Sogge ’02].
  - It is not clear how to directly allow for a boundary. Instead cutoff-type arguments [KSS ’00,’02] must be used.

Multiplier Approach

- Initiated by Rodnianski [’05] in the context of Strichartz estimates with angular regularity.
- Developed for variable coefficient wave equations and wave equations in exterior domains by M.-Sogge [’05].
- A similar estimate for variable coefficient wave equations (and wave equations on curved backgrounds) was obtained by Alinhac [’05], using (in part) the ghost weight method.
- The proof is quite similar to those used to prove local smoothing estimates for Schrödinger equations.

Multiplier Approach

- Contract the energy-momentum tensor

  $Q_{\alpha\beta}[u]=\partial_\alpha u\partial_\beta u-(1/2)m_{\alpha\beta}\partial^\gamma u\partial_\gamma u$

with the radial vector field $X=f(r)\partial_r$, and define the momentum density

  $P_\alpha[u,X]=Q_{\alpha\beta}[u]X^\beta$

- Computing its divergence,

$$D^\alpha P_\alpha[u,X]=f'(r)(\partial_r u)^2+\frac{f(r)}{r}|\nabla\!\!\!/ u|^2-\frac{1}{2}(tr\pi)\partial^\gamma u\partial_\gamma u-f(r)\partial_r u\Box u$$

$$tr\pi=f'(r)+(n-1)\frac{f(r)}{r}$$

Multiplier Approach

- In order to get a positive lower bound for the first three terms, we need better control on the Lagrangian term. To do so, we introduce a modified momentum density

$$P_\alpha[u,X]=Q_{\alpha\beta}[u]X^\beta+\frac{n-1}{2}\frac{f(r)}{r}u\partial_\alpha u-\frac{n-1}{4}\partial_\alpha\left(\frac{f(r)}{r}\right)u^2$$

Multiplier Approach

- Computing the divergence, we have

$$D^\alpha P_\alpha[u,X] = \frac{1}{2} f'(r)(\partial_r u)^2 + \frac{f(r)}{r}|\nabla\!\!\!/ u|^2 - \frac{1}{2} f'(r)|\nabla\!\!\!/ u|^2$$
$$+\frac{1}{2} f'(r)(\partial_t u)^2 - f(r)\partial_r u \Box u - \frac{n-1}{2}\frac{f(r)}{r} u\Box u - \frac{n-1}{4}\Delta\left(\frac{f(r)}{r}\right)u^2$$

- Integrate both sides in space and time, apply the divergence theorem, and for our choice of f the last term will be positive for $n \geq 3$

Multiplier Approach

- Choose $f(r) = \frac{r}{r+R}$, R constant.
- It follows that

$$\int\int \frac{R}{(r+R)^2}(\partial_r u)^2 dxdt + \int\int \frac{1}{r+R}|\nabla\!\!\!/ u|^2 dxdt + \int\int \frac{R}{(r+R)^2}(\partial_t u)^2 dxdt$$
$$\leq C\|u'(0,\cdot)\|_2 + C\int\int |\partial u \Box u| dxdt + C\int\int \left|\frac{u}{r+R}\Box u\right| dxdt$$

- Here, the energy inequality and a Hardy inequality are used to bound the (time) boundary terms

Multiplier Approach

- Moreover, if the spatial integral is instead taken over the exterior to a star-shaped obstacle, the resulting boundary terms have a favorable sign (as in [Morawetz, '61]), and thus the same bound holds.

Multiplier Approach

- We now choose $R=1$ and $R=2^j$ which respectively show that

$$\int \int_{|x|\leq 1} (\partial u)^2 dxdt$$

and

$$\int \int_{|x|\approx 2^j} \langle x\rangle^{-1} (\partial u)^2 dxds$$

are bounded by

$$C\|u'(0,\cdot)\|_2 + C\int\int |\partial u \Box u| dxdt + C\int\int \left|\frac{u}{r+R}\Box u\right| dxdt$$

Multiplier Approach

- Summing these estimates over j yields the desired result.
- Moreover, the proof shows that if the space-time gradient is replaced by only the angular derivatives, then the log term is unnecessary.
- Using standard arguments, the proof may be modified to yield a similar estimate for certain variable coefficient wave equations.

Application 1: Long time existence in exterior domains

- Using the variable coefficient KSS estimate, one can mimic the argument for semilinear equations outlined previously. Doing so, one can simplify the proof of almost global existence in $n=3$ of KSS ['02] and global existence in $n \geq 4$ of Shibata-Tsutsumi ['86] and M.-Sogge ['05] when the obstacle is star-shaped.

Application 2: Global existence for null-form wave equations

- With the variable coefficient KSS estimate, one can show, using only energy methods, that solutions to null-form wave equations exist globally in the exterior of star-shaped obstacles [M.-Sogge, '05].
- This is a simplification of previous results of KSS ['02], M.-Sogge ['05], M.-Nakamura-Sogge ['05].
- Such arguments (not relying on the fundamental solution) have been useful in studies of elasticity (in the boundaryless case). See, e.g., [Sideris-Tu, '01], [Sideris, '00], [Klainerman-Sideris, '96].

Application 2: Global existence for null-form wave equations

- In previous studies in exterior domains, the scaling vector field needed to be used with care. While it has bounded normal component on the boundary, its coefficients blow-up in any neighborhood of the obstacle. Thus, arguments were restricted to using relatively few occurrences of the scaling vector field.
- Using the variable coefficient KSS estimate, one can show that it is no longer necessary to distinguish between the scaling vector field and the other admissible (translations and spatial rotations) vector fields

Application 2: Global existence for null-form wave equations.

- This is done by introducing boundary condition preserving vector fields

$$\tilde{L} = t\partial_t + \eta(x) r \partial_r \qquad \tilde{\partial}_k = \eta(x)\partial_k,\ k = 1,2,3$$

$$\tilde{\Omega}_{ij} = \eta(x)\left(x_i\partial_j - x_j\partial_i\right),\ 1 \le i < j \le 3$$

as in M.-Sogge ['05]. The commutators are local terms which can be handled using elliptic regularity and induction.

Application 3: Dissipative wave equations

- [M., '05] In the multiplier approach, we "modified" the momentum density in order to add in additional Lagrangian terms.
- Before this was done to cancel some of the existing Lagrangian terms in the divergence of the momentum density.
- Here, we intentionally add in additional Lagrangian terms. This comes at the cost of a $(\partial_t u)^2$ term with a negative sign.

Application 3: Dissipative wave equations

- As one gets bounds for the space-time integral of $(\partial_t u)^2$ from the energy-estimates for dissipative equations, these terms pose no difficulty.
- Moreover, for Dirichlet boundary conditions, the momentum density and the modified momentum density have the same boundary terms.

Application 3: Dissipative wave equations

- By looking at the difference between the two momentum densities, one can get global existence in any exterior domain (regardless of the geometry), any interior domain, and waveguides ($\mathbb{R}^n \times \Omega$ where Ω is any compact domain) for $n \ge 3$ and for Dirichlet boundary conditions.
- The same argument also works for dissipative Klein-Gordon equations.

Application 3: Dissipative wave equations

- For Neumann boundary conditions, things are a bit more delicate. Results can be obtained for exterior domains with Neumann boundary conditions so long as
  - A nonlinear compatibility condition (as introduced in M.-Sogge-Stewart ['05] in the context of waveguides) holds so that energy methods can be applied.
  - The obstacle is star-shaped.

Application 4: Neumann wave equations in waveguides

- Ongoing work with C. Sogge and A. Stewart.
- Global existence for Dirichlet wave equations in waveguides was shown in [M.-Sogge-Stewart, '05].
- Neumann boundary conditions are much more delicate as the Laplacian of the compact base has a zero mode.
- Partial results were obtained in [M.-Sogge-Stewart, '05] for equations of the form $\Box u = Q(\nabla_x u)$
- In sufficiently high dimensions, dependence on $\partial_t u$ and u can be permitted.

Application 4: Neumann wave equations in waveguides

- The proof is similar to that of KSS ['00], but requires a KSS estimate for Klein-Gordon equations which is independent of the mass.
- Using Fourier methods, such an estimate can be obtained. However, the estimate is weaker for the $\partial_t u$ terms (although it is stronger for the u terms). In both cases, the weight is $\langle x\rangle^{-1}$ rather than $\langle x\rangle^{-1/2}$

Application 4: Neumann wave equations in waveguides

- By contracting with $X = f(r)\partial_r$ where $r^2 = x_1^2 + x_2^2 + \ldots + x_n^2$ and $(x_1,\ldots,x_n,x_{n+1},\ldots,x_{n+d}) \in \mathbb{R}^n \times \Omega$ one can establish KSS estimates in Neumann waveguides where the compact base is star-shaped.
- Similar arguments can be used to construct such estimates for variable coefficient wave equations, thus allowing us to extend the result of M.-Sogge-Stewart to quasilinear equations of the form

$$\Box u = Q(\nabla_x u, \nabla_x^2 u)$$

Application 4: Neumann wave equations in waveguides

- It is not clear how to get bounds for $\partial_t u$ or for $\partial_j,\quad j = n+1,\ldots,n+d$ as both terms appear in the Lagrangian but with opposite sign.

Application 5: Klein-Gordon equations in exterior domains

- Ongoing work with C. Sogge and A. Stewart
- As mentioned previously, Fourier methods have not yielded a KSS estimate for $\partial_t u$ that is of the same strength of that for the wave equation.
- A problem also occurs with these terms using multiplier methods. The problem resembles that for waveguides in that the Lagrangian contains both $\partial_t u$ and u but with opposite signs.

Application 5: Klein-Gordon equations in exterior domains

- Arguing similarly to before (but modifying in such a way as to cancel all of the Lagrangian terms) yields a KSS estimate

$$\left\| \langle x \rangle^{-1/2-} \nabla_x u \right\|_{L_t^2 L_x^2} \le C \sum_{|\alpha| \le 1} \left\| \partial^\alpha u(0,\cdot) \right\|_2 + C \int_0^T \left\| (\Box + m^2) u(s,\cdot) \right\|_2 ds$$

- It is not currently known if the argument can be modified to give estimates for, e.g., the time derivative.

Application 5: Klein-Gordon equations in exterior domains.

- The KSS estimate for the spatial derivatives suffices to give small data global existence for

$$(\Box + m^2) u = Q(\nabla_x u, \nabla_x^2 u)$$

in the exterior of a star-shaped obstacle in dimensions $n \ge 4$.

- Based on known results in free space, this is far from the expected sharp result.

Application 6: Elasticity in exterior domains

- Ongoing work with C. Sogge and B. Thomases
- It would be desirable to try to establish a KSS estimate for the linearized equation of elasticity

$$\partial_t^2 u - c_2^2 \Delta u - (c_1^2 - c_2^2) \nabla(\nabla \cdot u)$$

- For the constant coefficient equation, it appears that such a bound can be obtained from those known for multiple speed systems of wave equations and from a Helmholtz-Hodge decomposition.

Application 6: Elasticity in exterior domains

- From the multiplier point of view (in order to obtain a variable coefficient KSS estimate), the nondiagonal terms appear to be preventing the establishment of a proper positive lower bound for the divergence of the momentum density.