Sharp Local Smoothing for Warped Product Manifolds with Smooth Inflection Transmission

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ABSTRACT. We consider a family of rotationally symmetric, asymptotically Euclidean manifolds with two trapped sets, one of which is unstable and one of which is semistable. We prove a sharp local smoothing estimate for the linear Schrödinger equation with a loss that depends on how flat the manifold is near each of the trapped sets. The result interpolates amongst the members of the family of similar estimates in [CW13]. As a consequence of the techniques used in our proof, we also show a sharp high-energy resolvent estimate with a polynomial loss that depends on how flat the manifold is near each of the trapped sets.

1. INTRODUCTION

In this paper, we study the local smoothing effect for the Schrödinger equation on a class of warped product manifolds with a trapped set that is mixed unstable and *semistable*, which we call inflection-transmission due to its relation to classical transmission problems (see Remark 1.5). Our main result is a generalization of the local smoothing estimate

$$\int_0^T \|\langle x\rangle^{-1/2-} e^{it\Delta} u_0\|_{H^{1/2}}^2 \,\mathrm{d}t \lesssim \|u_0\|_{L^2}^2.$$

Such estimates first appeared in [CS88], [Sjö87], and [Veg98], and were extended to nontrapping asymptotically flat geometries in [CKS95] and [Doi96a] (see, e.g., [RT07], [MMT08] for some recent generalizations). The presence of trapping necessitates a loss of smoothing as was shown in [Doi96b]. Situations in which

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the trapping is unstable and nondegenerate have already been studied in [Bur04], [Chr07, Chr08, Chr11], [Dat09], and [BGH10], amongst several others. Trapping that is unstable but degenerately so was the topic of [CW13]. The novel thing in this paper is the existence of semistable trapping, that is, trapping which is stable from one direction and unstable from another direction.

Let us begin by describing the geometry. We shall demonstrate the effect of this sort of trapping by examining an explicit example. Let m_1 and m_2 be positive integers, and set

$$A^2(x) = 1 + \int_0^x \frac{y^{2m_1-1}(y-1)^{2m_2}}{(1+y^2)^{m_1+m_2-1}}\,\mathrm{d}y.$$

As the integrand in the last term satisfies

$$\frac{x^{2m_1-1}(x-1)^{2m_2}}{(1+x^2)^{m_1+m_2-1}} \sim \begin{cases} x^{2m_1-1}, & x \sim 0, \\ \dfrac{(x-1)^{2m_2}}{2^{m_1+m_2-1}} & x \sim 1, \\ x, & |x| \to \infty, \end{cases}$$

we notice that

$$A^2(x) \sim \begin{cases} 1 + x^{2m_1}, & x \sim 0, \\ C_1 + c_2(x-1)^{2m_2+1} & x \sim 1, \\ x^2, & |x| \to \infty. \end{cases} \tag{1.1}$$

Here, $C_1 > 1$ and $c_2 < 1$ are constants which are easily computed but inessential, except for their relative sizes compared to 1. As will be clear in the sequel, the specific structure of A is inessential, and only the location and nature of the critical points and behavior at infinity matter.

Now, let $X = \mathbb{R}_x \times \mathbb{R}_\theta / 2\pi\mathbb{Z}$, equipped with the metric

$$\mathrm{d}s^2 = \mathrm{d}x^2 + A^2(x)\,\mathrm{d}\theta^2,$$

so that X is asymptotically Euclidean with two ends and has two trapped sets. The trapping occurs where $A'(x) = 0$, which is at $x = 0$ and $x = 1$, respectively (see Figure 1.1). The metric determines the volume form

$$\mathrm{d}\,\mathrm{Vol} = A(x)\,\mathrm{d}x\,\mathrm{d}\theta$$

and the Laplace-Beltrami operator acting on 0-forms

$$\Delta f = (\partial_x^2 + A^{-2}\,\partial_\theta^2 + A^{-1}A'\,\partial_x)f.$$

More general warped product manifolds (with arbitrary compact cross section) can also be considered (see Remark 1.4 and [Chr13]); however, for simplicity in exposition, we consider only the surface of revolution case here.

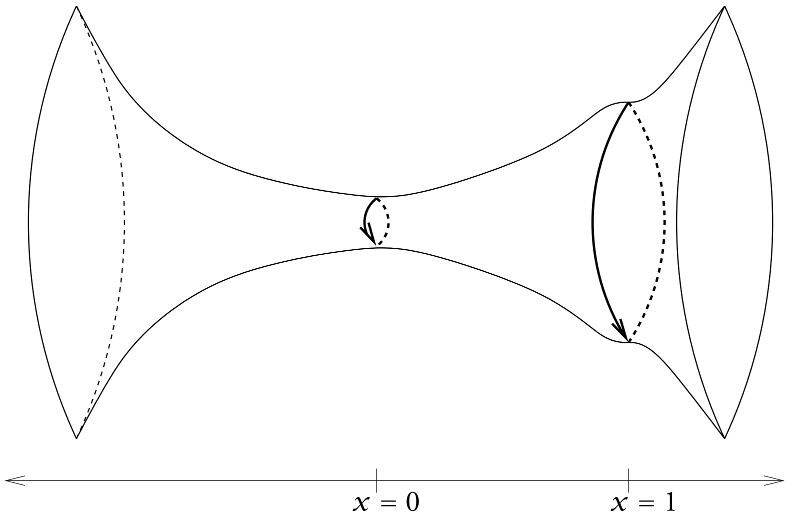

FIGURE 1.1. A piece of the manifold X with the trapped sets at $x = 0$ and at $x = 1$

Our main result is the following local smoothing estimate with sharp loss. Using the common notation $\mathrm{D}_t = (1/i)\,\partial_t$, we have the following theorem.

Theorem 1.1 (Local Smoothing). *Suppose X is as above with $m_1, m_2 \geqslant 1$, and assume u solves*

$$\begin{cases} (\mathrm{D}_t - \Delta)u = 0 & \text{in } \mathbb{R} \times X, \\ u\big|_{t=0} = u_0 \in H^s, \end{cases}$$

for some $s > 0$ sufficiently large. Then, for any $T < \infty$, there exists a constant $C_T > 0$ such that

$$\begin{aligned} \int_0^T \big(\|\langle x\rangle^{-1}\,\partial_x u\|^2_{L^2(\mathrm{d\,Vol})} + \|\langle x\rangle^{-3/2}\,\partial_\theta u\|^2_{L^2(\mathrm{d\,Vol})}\big)\,\mathrm{d}t \\ \leqslant C_T\big(\|\langle \mathrm{D}_\theta\rangle^{\beta(m_1,m_2)} u_0\|^2_{L^2(\mathrm{d\,Vol})} + \|\langle \mathrm{D}_x\rangle^{1/2} u_0\|^2_{L^2(\mathrm{d\,Vol})}\big), \end{aligned}$$

where

$$\beta(m_1, m_2) = \max\left(\frac{m_1}{m_1 + 1}, \frac{2m_2 + 1}{2m_2 + 3}\right). \tag{1.2}$$

Moreover, this estimate is sharp, in the sense that no polynomial improvement in regularity is true.

This theorem requires some remarks.

Remark 1.2. Observe that the maximum amount of smoothing in the presence of inflection-transmission trapping is a *gain* in regularity of $2/(2m_2 + 3)$

derivatives. Each of these fractions lies between sequential fractions in the numerology of [CW13], since

$$\frac{1}{(m+1)+1} < \frac{2}{2m+3} < \frac{1}{m+1}.$$

Remark 1.3. In the theorem above, the weights at infinity are different than those that appear in the standard Euclidean estimate. Standard cutoff arguments would allow us to make these match. The key new aspect of the theorem, however, is the behavior near the trapped sets; and for clarity in the proof, we do not modify the weights.

Remark 1.4. Theorem 1.1, and indeed also Theorem 1.6 below, are of course true in many more situations. In our example chosen to illustrate the effect of such trapping, we are working on a surface of revolution where the generating curve is parametrized by arclength. The techniques herein apply more generally to warped-product manifolds. In particular, $\mathbb{R}_\theta/2\pi\mathbb{Z}$ may be replaced by a compact manifold of any dimension. The inflection-transmission trapping is characterized by an inflection point in the potential that results from a reduction to a one-dimensional problem via an eigenvalue decomposition of the compact portion of the manifold.

Of particular interest, the microlocalization step which separates the trapped sets at different energies used to prove (2.8) indicates the same result applies to a manifold with one Euclidean end and only an inflection transmission trapped set. On the other hand, if our manifold has two Euclidean ends, a degenerate hyperbolic trapped set, and *two* inflection transmission trapped sets *at the same* semiclassical energy, it is natural to suspect that such a theorem is no longer true because the two inflection transmission sets must tunnel to each other. However, it is easy to see that the theorem still applies in this case, since the stable/unstable manifolds for the degenerate hyperbolic trapped set form a separatrix (in other words, the degenerate hyperbolic trapped set is at *higher* semiclassical energy). Hence, the same microlocalization applies, and so does the theorem.

Remark 1.5. We briefly discuss why we have chosen to call this type of smooth trapping "inflection-transmission" type trapping. The *inflection* label refers to the fact that the effective potential after separating variables has an inflection point at the trapped set; moreover, we use *transmission* because this kind of trapping bears some resemblance to the traditional transmission problem.

The traditional transmission problem concerns a wave equation in a medium for which the speed of propagation is distinct in different regions. For example, one might study solutions to the equation

$$\begin{cases} (\partial_t^2 - \Delta)u = 0 & \text{for } |x| < 1, \\ (\partial_t^2 - c^2\Delta)u = 0 & \text{for } |x| > 1, \end{cases}$$

where $c \neq 1$. Of course, one also needs to indicate appropriate boundary conditions at the interaction surface where $|x| = 1$ (see, e.g., [CPV01, CPV99, CV10]).

On the other hand, if we consider a surface of revolution given by a specific generating curve C in the (x_1, x_3) plane, rotated around the x_3 axis, we get a similar looking picture. Let

$$C = \{(x_1, 0, x_3) = (A(r), 0, B(r)) : r \geqslant 0\},$$

where

$$A(r) = \begin{cases} r, & \text{for } 0 \leqslant r \leqslant 1, \\ \frac{1}{2}r, & \text{for } r \geqslant 3, \end{cases}$$

and assume $0 \leqslant A'(r) \leqslant 1$ and A has an inflection point at, say, $r = 2$. The function $A(r)$ is sketched schematically in Figure 1.2.

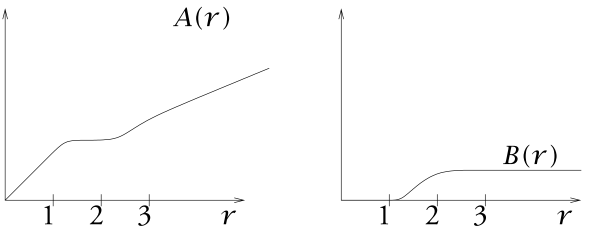

FIGURE 1.2. The functions $A(r)$ and $B(r)$

We suppose that B' is compactly supported in the region $1 \leq r \leq 3$, and we fix $B(0) = 0$. The function $B(r)$ is also depicted in Figure 1.2.

Rotating the curve C about the x_3 axis in $\mathbb{R}^3$ yields a manifold which is flat near 0 and flat outside a compact set, and which changes "height" in between (see Figure 1.3).

Moreover, if we compute the Laplacian on this surface of revolution, we see that

$$\Delta_g = \frac{1}{r}\,\partial_r r\,\partial_r + \frac{1}{r^2}\,\partial_\theta^2, \qquad 0 \leq r \leq 1,$$

but that

$$\Delta_g = 4\left(\frac{1}{r}\,\partial_r r\,\partial_r + \frac{1}{r^2}\,\partial_\theta^2\right), \quad r \geq 3.$$

See, for example, [Boo11], where such a computation has been carried out in much detail.

As in [CW13], once we prove Theorem 1.1, we can obtain a resolvent bound. For simplicity, say that our surface of revolution is Euclidean at infinity. That is, assume $A(x) = x$ for $|x| \gg 0$. Alternatively, we could require dilation analyticity at infinity, which would permit asymptotically conic spaces such as were treated in [WZ00].

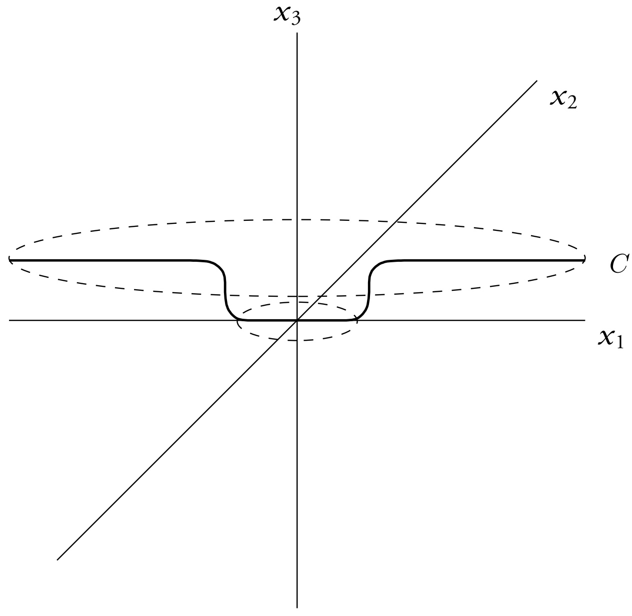

FIGURE 1.3. The manifold obtained by rotating the curve C about the x_3 axis

We now let $R(\lambda) = (-\Delta_g - \lambda^2)^{-1}$ denote the resolvent on X (where it exists), and take $\operatorname{Im}\lambda < 0$ as our physical sheet. With a choice of appropriate branch cut, $\chi R(\lambda)\chi$ extends meromorphically to $\{\lambda \in \mathbb{R} : \lambda \gg 0\}$ for any $\chi \in C_c^\infty(X)$ (see, e.g., [SZ91]). In the degenerate inflection point setting, we have the following result.

Theorem 1.6. *For any $\chi \in C_c^\infty(X)$, there exists a constant $C = C_{m_1,m_2,\chi} > 0$ such that, for $\lambda \gg 0$,*

$$\|\chi R(\lambda - i0)\chi\|_{L^2\to L^2} \leqslant C \max\{\lambda^{-2/(m_1+1)}, \lambda^{-4/(2m_2+3)}\}.$$

Moreover, this is the nut estimate, in the sense that no better polynomial rate of decay holds true for every cutoff χ.

2. LOCAL SMOOTHING ESTIMATES

In this section, we prove the main local smoothing estimate. In the subsequent section, we shall saturate the inequality, thus showing that the loss is sharp. The proof of the estimate uses a positive commutator argument. On Euclidean space, such a proof of local smoothing is well known, though the interested reader can see [CW13, Section 2.1] for an exposition quite akin to what follows.

We first conjugate Δ and reduce the problem to be one-dimensional. Indeed, we set $L : L^2(X, \mathrm{d}\,\mathrm{Vol}) \to L^2(X, \mathrm{d}x\, \mathrm{d}\theta)$ to be the isometry

$$Lu(x, \theta) = A^{1/2}(x)u(x, \theta).$$

With mild assumptions on A, $\tilde{\Delta} = L\Delta L^{-1}$ is (essentially) self-adjoint on $L^2(X, \mathrm{d}x\, \mathrm{d}\theta)$. More explicitly, we have

$$-\tilde{\Delta} f = (-\partial_x^2 - A^{-2}(x)\, \partial_\theta^2 + V_1(x))f$$

with

$$V_1(x) = \frac{1}{2}A''A^{-1} - \frac{1}{4}(A')^2A^{-2}.$$

Given a function ψ on X, we expand it into its Fourier series, $\psi(x, \theta) = \sum_k \varphi_k(x)e^{ik\theta}$, and note that

$$(-\tilde{\Delta} - \lambda^2)\psi = \sum_k e^{ik\theta}(P_k - \lambda^2)\varphi_k(x),$$

where

$$P_k\varphi_k(x) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2A^{-2}(x) + V_1(x)\right)\varphi_k(x).$$

By setting $h = k^{-1}$, we pass to the semiclassical operator

$$(P(h) - z)\varphi(x) = \left(-h^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) - z\right)\varphi(x),$$

where the potential is $V(x) = A^{-2}(x) + h^2V_1(x)$ and the spectral parameter is $z = h^2\lambda^2$.

We first show the following result.

Proposition 2.1. *Suppose u solves*

$$\begin{cases} (\mathrm{D}_t - \tilde{\Delta})u = 0, \\ u(0, x, \theta) = u_0. \end{cases} \tag{2.1}$$

Then, for any $T < \infty$, there exists a constant $C_T > 0$ such that

$$\begin{aligned}\int_0^T (\|\langle x\rangle^{-1}\, \partial_x u\|^2_{L^2(\mathrm{d}x\,\mathrm{d}\theta)} + \|\langle x\rangle^{-3/2}\, \partial_\theta u\|^2_{L^2(\mathrm{d}x\,\mathrm{d}\theta)})\, \mathrm{d}t \\ \leqslant C_T(\|\langle \mathrm{D}_\theta\rangle^{\beta(m_1,m_2)}u_0\|^2_{L^2(\mathrm{d}x\,\mathrm{d}\theta)} + \|\langle \mathrm{D}_x\rangle^{1/2}u_0\|^2_{L^2(\mathrm{d}x\,\mathrm{d}\theta)}),\end{aligned}$$

where $\beta(m_1, m_2)$ is as in (1.2).

The equation (2.1) is obtained by conjugating the original equation by the operator L. Upon conjugating back, Proposition 2.1 shows that the estimate of Theorem 1.1 holds.

2.1. Proof of Proposition 2.1. The proof will be broken into three steps. The first uses a positive commutator argument to prove full smoothing away from the periodic orbits at $x = 0$ and $x = 1$. We then expand the solution into a Fourier series to reduce to a one-dimensional problem, and we reduce the problem to understanding the high-frequency asymptotics of the solution. Using a TT^* argument, gluing techniques, and a semiclassical rescaling, we show that the high-frequency estimate follows from a cutoff resolvent estimate near each instance of trapping; we subsequently prove those.

2.1.1. The estimate away from $x = 0$ ***and*** $x = 1$ For a self-adjoint operator $\tilde{\Delta}$ and a time-independent, self-adjoint multiplier B, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u, Bu\rangle = -2\,\mathrm{Im}\langle(\mathrm{D}_t - \tilde{\Delta})u, Bu\rangle + i\langle[-\tilde{\Delta}, B]u, u\rangle. \tag{2.2}$$

In particular, if $B = \frac{1}{2}\arctan(x)\mathrm{D}_x + \frac{1}{2}\mathrm{D}_x\arctan(x)$, then

$$i[-\tilde{\Delta}, B] = 2\mathrm{D}_x\langle x\rangle^{-2}\mathrm{D}_x + 2\mathrm{D}_\theta A'A^{-3}\arctan(x)\mathrm{D}_\theta - \frac{3x^2-1}{\langle x\rangle^6} - V_1'\arctan(x).$$

Upon integrating (2.2) over $[0, T]$, we obtain

$$\int_0^T \langle i[-\tilde{\Delta}, B]u, u\rangle\,\mathrm{d}t = i\langle u, \arctan(x)\,\partial_x u\rangle\,|_0^T - \frac{i}{2}\langle u, \langle x\rangle^{-1}u\rangle\,|_0^T$$

for a solution u to (2.1). Using energy estimates, the right side is controlled by $\|u_0\|_{H^{1/2}}^2$. Noting also that energy estimates permit the control

$$\begin{aligned}\Big|\int_0^T \Big\langle \frac{3x^2-1}{\langle x\rangle^6}u + V_1'\arctan(x)u, u\Big\rangle\,\mathrm{d}t\Big| \\ \leqslant CT\sup_{0\le t\le T}\|u(t,\cdot)\|_{L^2}^2 \leqslant C_T\|u_0\|_{H^{1/2}}^2,\end{aligned}$$

it now follows from integration by parts that we have established

$$\int_0^T (\|\langle x\rangle^{-1}\,\partial_x u\|_{L^2}^2 + \|\sqrt{A'A^{-3}\arctan(x)}\,\partial_\theta u\|_{L^2}^2)\,\mathrm{d}t \leqslant C_T\|u_0\|_{H^{1/2}}^2.$$

We observe that $A'A^{-3}\arctan(x) \geqslant 0$ and satisfies

$$A'A^{-3}\arctan(x) \sim \begin{cases} x^{2m_1}, & x \sim 0, \\ c_2'(x-1)^{2m_2}, & x \sim 1, \\ |x|^{-3}, & |x| \to \infty. \end{cases}$$

Thus,

$$\| |x|^{m_1} |x-1|^{m_2} \langle x\rangle^{-m_1-m_2-3/2} \partial_\theta u\|_{L^2} \leqslant C\|\sqrt{A' A^{-3} \arctan(x)}\, \partial_\theta u\|_{L^2},$$

and hence we have the estimate

$$\int_0^T (\|\langle x\rangle^{-1} \partial_x u\|_{L^2}^2 + \| |x|^{m_1} |x-1|^{m_2} \langle x\rangle^{-m_1-m_2-3/2} \partial_\theta u\|_{L^2}^2)\, \mathrm{d}t \leqslant C_T \|u_0\|_{H^{1/2}}^2. \tag{2.3}$$

That is, we have perfect smoothing in the radial direction and in the θ direction away from $x = 0$ and $x = 1$, which is precisely where the trapped sets reside.

2.1.2. Fourier decomposition To get an estimate in the directions tangential to the trapping, we decompose the solution into Fourier series

$$u(t,x,\theta) = \sum_k e^{ik\theta} u_k(t,x) \quad \text{and} \quad u_0(x,\theta) = \sum_k e^{ik\theta} u_{0,k}(x).$$

By Plancherel's theorem, it suffices to show

$$\int_0^T (\|\langle x\rangle^{-1} \partial_x u_k\|_{L^2(\mathrm{d}x)}^2 + k^2 \|\langle x\rangle^{-3/2} u_k\|_{L^2(dx)}^2)\, \mathrm{d}t \leqslant C_T (\|\langle k\rangle^{\beta(m_1,m_2)} u_{0,k}\|_{L^2(\mathrm{d}x)}^2 + \|\langle \mathrm{D}_x\rangle^{1/2} u_{0,k}\|_{L^2(\mathrm{d}x)}^2).$$

We note that, as $\partial_\theta u_0 = 0$, where in an abuse of notation u_0 here stands for the zero mode, the estimate when $k = 0$ follows trivially from (2.3). Thus, it remains to show that

$$\int_0^T \|\chi(x) k u_k\|_{L^2(\mathbb{R})}^2\, \mathrm{d}t \leqslant C_T (\|\langle k\rangle^{\beta(m_1,m_2)} u_{0,k}\|_{L^2}^2 + \|u_{0,k}\|_{H^{1/2}}^2), \quad |k| \geqslant 1$$

for some $\chi \in C_c^\infty(\mathbb{R})$ with $\chi(x) \equiv 1$ in a neighborhood of $x = 0$ and also in a neighborhood of $x = 1$.

In the sequel, we shall be working with a fixed k, and as such, shall drop the subscript notation. Set

$$P_k = \mathrm{D}_x^2 + A^{-2}(x) k^2 + V_1(x).$$

Notice that P_k is merely $-\tilde{\Delta}$ applied to the kth mode. We fix an even function $\psi \in C_c^\infty(\mathbb{R})$ which is 1 for $|r| \leq \varepsilon$ and vanishes for $|r| \geq 2\varepsilon$, where $\varepsilon > 0$ will be determined later. Then, let

$$u = u_{\mathrm{hi}} + u_{\mathrm{lo}}, \quad u_{\mathrm{hi}} = \psi(\mathrm{D}_x/k) u.$$

2.1.3. Low-frequency estimate. We first examine u_{lo}, and reduce the estimation of it to the estimation of a term similar to u_{hi}. We also observe that $u_{\mathrm{lo}} = (1 - \psi(\mathrm{D}_x/k))u$ solves

$$(\mathrm{D}_t + P_k)u_{\mathrm{lo}} = -[P_k, \psi(\mathrm{D}_x/k)]u = k\langle x\rangle^{-1}L_k\langle x\rangle^{-2}\tilde{\psi}(\mathrm{D}_x/k)u,$$

where L_k is L^2 bounded uniformly in k, and where $\tilde{\psi} \in C_c^\infty$ which is equal to one on the support of ψ.

Choosing the same multiplier B, replacing $-\tilde{\Delta}$ with the self-adjoint P_k, and integrating (2.2) yields

$$\begin{aligned}(2.4)\quad &\left|\int_0^T \langle [P_k, B]u_{\mathrm{lo}}, u_{\mathrm{lo}}\rangle\,\mathrm{d}t\right| \\ &\quad\leqslant C\bigg(|\langle u_{\mathrm{lo}}, \arctan(x)\,\partial_x u_{\mathrm{lo}}\rangle|\,\Big|_0^T + |\langle u_{\mathrm{lo}}, \langle x\rangle^{-1}u_{\mathrm{lo}}\rangle|\,\Big|_0^T \\ &\qquad\qquad + \left|\int_0^T \langle\langle x\rangle^{-1}kL_k\langle x\rangle^{-2}\tilde{\psi}(\mathrm{D}_x/k)u, Bu_{\mathrm{lo}}\rangle\,\mathrm{d}t\right|\bigg).\end{aligned}$$

Continuing to argue as above shows that

$$\begin{aligned}&\int_0^T \|\langle x\rangle^{-1}\,\partial_x u_{\mathrm{lo}}\|_{L^2}^2\,\mathrm{d}t \\ &\quad\leqslant C_T\bigg(\|u_0\|_{H^{1/2}}^2 + \left|\int_0^T \langle\langle x\rangle^{-1}kL_k\langle x\rangle^{-2}\tilde{\psi}(\mathrm{D}_x/k)u, Bu_{\mathrm{lo}}\rangle\,\mathrm{d}t\right|\bigg).\end{aligned}$$

Applying the Schwarz inequality to the last term, and bootstrapping, we obtain

$$\begin{aligned}(2.5)\quad &\int_0^T \|\langle x\rangle^{-1}\,\partial_x u_{\mathrm{lo}}\|_{L^2}^2\,\mathrm{d}t \\ &\quad\leqslant C_T\bigg(\|u_0\|_{H^{1/2}}^2 + \int_0^T \|k\langle x\rangle^{-2}\tilde{\psi}(\mathrm{D}_x/k)u\|_{L^2}^2\,\mathrm{d}t\bigg).\end{aligned}$$

The frequency cutoff guarantees that

$$\int_0^T \|\langle x\rangle^{-1}ku_{\mathrm{lo}}\|_{L^2}^2\,\mathrm{d}t \leqslant C\int_0^T \|\langle x\rangle^{-1}\,\partial_x u_{\mathrm{lo}}\|_{L^2}^2\,\mathrm{d}t.$$

As (2.3) provides control on the last term in (2.5) away from $x = 0$ and $x = 1$, it suffices to prove

$$\int_0^T \|\chi k\tilde{\psi}(\mathrm{D}_x/k)u\|_{L^2}^2\,\mathrm{d}t \leqslant C_T\|k^{\beta(m_1,m_2)}u_0\|_{L^2}^2.$$

Here, χ is a cutoff which is 1 in a neighborhood of the trapped geodesics at $x = 0$ and $x = 1$. The desired bound, then, will follow once u_{hi} is controlled, as the precise choice of cutoff ψ is inessential.

2.1.4. High-frequency estimate. It remains to estimate u_{hi} in the vicinity of $x = 0$ and $x = 1$. We fix a cutoff $\chi \in C_c^\infty(\mathbb{R})$ which is 1 in a neighborhood of $x = 0$ and in a neighborhood of $x = 1$. Let

$$F(t)g = \chi(x)\psi(\mathrm{D}_k/k)k^r e^{-itP_k}g,$$

where the constant $r > 0$ will be determined later. We seek to determine r so that $F : L_x^2 \to L^2([0,T]; L_x^2)$. The resulting inequality,

$$\|k^{1-r}F(t)u_0\|_{L^2([0,T];L_x^2)} \leqslant C_T\|k^{1-r}u_0\|_{L^2}, \tag{2.6}$$

is a local smoothing estimate. Also, F is such a mapping if and only if $FF^* : L^2L^2 \to L^2L^2$, where we have abbreviated $L^2([0,T]; L_x^2) = L^2L^2$. A straightforward computation shows that

$$FF^*f(t,x) = \chi(x)\psi(\mathrm{D}_x/k)k^{2r}\int_0^T e^{-i(t-s)P_k}\psi(\mathrm{D}_x/k)\chi(x)f(s,x)\,\mathrm{d}s,$$

and

$$\|FF^*f\|_{L^2L^2} \leqslant C_T\|f\|_{L^2L^2}$$

is the desired estimate. We write $FF^*f(t,x) = \chi(x)\psi(\mathrm{D}_x/k)(v_1 + v_2)$, where

$$v_1 = k^{2r}\int_0^t e^{-i(t-s)P_k}\psi(\mathrm{D}_x/k)\chi(x)f(s,x)\,\mathrm{d}s,$$
$$v_2 = k^{2r}\int_t^T e^{-i(t-s)P_k}\psi(\mathrm{D}_x/k)\chi(x)f(s,x)\,\mathrm{d}s.$$

Thus,

$$(\mathrm{D}_t + P_k)v_\ell = (-1)^\ell ik^{2r}\psi(\mathrm{D}_x/k)\chi(x)f, \quad \ell = 1, 2,$$

and $\|\chi\psi v_\ell\|_{L^2L^2} \leqslant C_T\|f\|_{L^2L^2}$ would imply the desired estimate. By Plancherel's theorem, this is equivalent to showing that

$$\|\chi\psi\hat{v}_\ell\|_{L^2L^2} \leqslant C_T\|\hat{f}\|_{L^2L^2},$$

where $\hat{f}$ denotes the Fourier transform of f in the time variable. That is, we are required to show that, uniformly in τ,

$$\|\chi\psi k^{2r}(\tau \pm i0 + P_k)^{-1}\psi\chi\|_{L_x^2\to L_x^2} = O(1).$$

Setting, as above, $-z = \tau k^{-2}$, $h = k^{-1}$, and $V = A^{-2}(x) + h^2V_1(x)$, we need

$$\begin{aligned}\|\chi(x)\psi(h\mathrm{D}_x)(-z \pm i0 + (h\mathrm{D}_x)^2 + V)^{-1}\psi(h\mathrm{D}_x)\chi(x)\|_{L^2\to L^2}\\ \leqslant Ch^{-2(1-r)}.\end{aligned} \tag{2.7}$$

We recall that $P = (h\,\mathrm{D}_x)^2 + V$, and we shall use gluing techniques to reduce the proving of

$$\|\rho_{-s}(P - z)u\|_{L^2} \geqslant ch^{-2\beta(m_1,m_2)}\|\rho_s u\|_{L^2}, \quad s < -\frac{1}{2} \tag{2.8}$$

(which implies (2.7) with $1 - r = \beta(m_1, m_2)$ as desired) to the proving of microlocal invertibility estimates near the trapped sets. In (2.8), ρ_s is a smooth function that is equal to one on a large compact set and is equivalent to $\langle x\rangle^s$ near infinity.

The gluing techniques we shall employ are outlined in [Chr13] (see also [Chr08, Proposition 2.2] and [DV12]).

Recall that we are working in $T^*\mathbb{R}$ with principal symbol $p = \xi^2 + V(x)$ where the potential $V(x)$ is a short range perturbation of x^{-2} and has critical points at precisely $x = 0, 1$. The critical point at $x = 0$ is a maximum with value 1, while the critical point at 1 is an inflection point with potential value C_1^{-1}. This means that, in terms of the Hamiltonian vector field H_p, the level set $\{p = 1\}$ contains the critical point $(0, 0)$, and the level set $\{p = C_1^{-1}\}$ contains the critical point $(1, 0)$. Furthermore, $\pm V'(x) \le 0$ for $\pm x \ge 0$ with equality only at these critical points.

As in [Chr13], we fix a few cutoffs. Let $M > 1$ be sufficiently large so that there is a symbol p_0 such that $p_0 = p$ for $|x| \ge M - 1$ and the operator P_0 associated to symbol p_0 satisfies

$$\|\rho_{-s}(P_0 - z)u\|_{L^2} \geqslant c\frac{h}{\log(1/h)}\|\rho_s u\|_{L^2}.$$

Such a P_0 is, for example, the $m = 1$ case of [CW13], and such bounds follow from [Chr07, Chr11]. Here, ρ_s is a smooth function such that $\rho_s > 0$, $\rho_s(x) \equiv 1$ on a neighborhood of $\{|x| \le 2M\}$, and $\rho_s \equiv \langle x\rangle^s$ for x sufficiently large. We choose $\Gamma \in C_c^\infty(\mathbb{R})$ with $\Gamma \equiv 1$ on $\{|x| \le M - 1\}$ with support in $\{|x| \le M\}$. In particular, $p = p_0$ on $\operatorname{supp}(1 - \Gamma)$.

For some fixed $\varepsilon_0 > 0$, we let

$$\Lambda(r) := \int_0^r \langle t\rangle^{-1-\varepsilon_0}\,\mathrm{d}t, \tag{2.9}$$

which is a function chosen to be globally bounded with positive derivative and $\Lambda(r) \sim r$ near $r = 0$. Then, let $a(x, \xi) = \Lambda(x)\Lambda(\xi)$, so that

$$H_p a = (2\xi\,\partial_x - V'(x)\,\partial_\xi)a = 2\xi\Lambda(\xi)\Lambda'(x) - V'(x)\Lambda(x)\Lambda'(\xi).$$

Since $\pm V'(x) < 0$ for $\pm x > 0$, $x \ne 1$, for any $\varepsilon > 0$ we have

$$H_p a \ge c_0 > 0, \quad |x| \in [\varepsilon/2, 1 - \varepsilon/2] \cup [1 + \varepsilon/2, M]. \tag{2.10}$$

We further have

$$H_p a \ge c_0' > 0, \quad |\xi| \ge \delta > 0 \text{ and } |x| \le M. \tag{2.11}$$

For $j = 0, 1$, we let $\Gamma_j(x)$ be equal to 1 for $|x - j| \leqslant \varepsilon/2$ with support in $\{|x - j| \leqslant \varepsilon\}$, and we set $\Gamma_2 = \Gamma - \Gamma_0 - \Gamma_1$ so that Γ_2 is supported where $|x| \in [\varepsilon/2, 1 - \varepsilon/2] \cup [1 + \varepsilon/2, M]$.

We may use a commutator argument to prove the necessary microlocal black box estimate for Γ_2. Indeed, for any $z \in \mathbb{R}$, using (2.10), we have

$$\begin{aligned} &2\,\mathrm{Im}\langle (P - z)\Gamma_2 u, a^w \Gamma_2 u\rangle \\ &\qquad = -i\langle (P - z)\Gamma_2 u, a^w \Gamma_2 u\rangle + i\langle a^w \Gamma_2 u, (P - z)\Gamma_2 u\rangle \\ &\qquad = i\langle [P, a^w]\Gamma_2 u, \Gamma_2 u\rangle \geqslant c_1 h\langle \Gamma_2 u, \Gamma_2 u\rangle \end{aligned}$$

for some $c_1 > 0$. Then,

$$c_1 h\|\Gamma_2 u\|^2 \leq 2\|(P - z)\Gamma_2 u\|\,\|a^w \Gamma_2 u\| \leq C\|(P - z)\Gamma_2 u\|\,\|\Gamma_2 u\|,$$

so that $\|\Gamma_2 u\| \leq C' h^{-1}\|(P - z)\Gamma_2 u\|$, as desired.

We now choose two microlocal cutoffs. For $j = 0, 1$, let $\psi_j = \psi_j(\xi^2 + V(x))$ be functions of the principal symbol p. For some $\delta > 0$ to be fixed momentarily, assume $\psi_0 \equiv 1$ for $\{|p - 1| \leqslant \delta\}$ with slightly larger support, and similarly assume $\psi_1 \equiv 1$ for $\{|p - C_1^{-1}| \leqslant \delta\}$ with slightly larger support. The parameter $\delta > 0$ may now be fixed, depending on $\varepsilon > 0$, so that $\psi_1 \equiv 1$ on $\mathrm{supp}(\Gamma_1) \cap \{\xi \equiv 0\}$. These cutoffs are depicted in Figure 2.1. Repeating the commutator argument above, but instead using (2.11), allows us to conclude that

$$\|\Gamma_0(1 - \psi_0)u\| \leq Ch^{-1}\|(P - z)\Gamma_0(1 - \psi_0)u\|$$

and

$$\|\Gamma_1(1 - \psi_1)u\| \leq Ch^{-1}\|(P - z)\Gamma_1(1 - \psi_1)u\|.$$

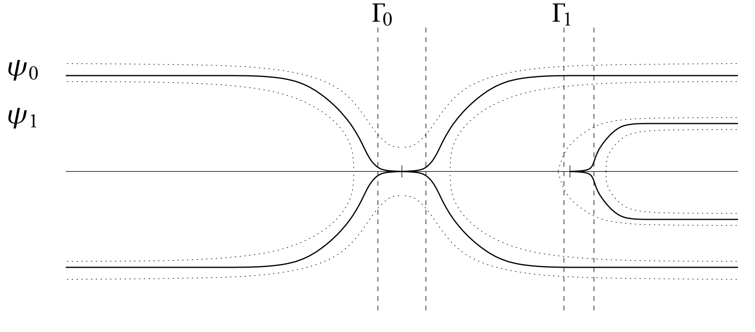

FIGURE 2.1. The cutoff functions used to apply [Chr13, Appendix].

We include a separate figure (Figure 2.2) that illustrates that such a microlocalization can be carried out in the case described in Remark 1.4.

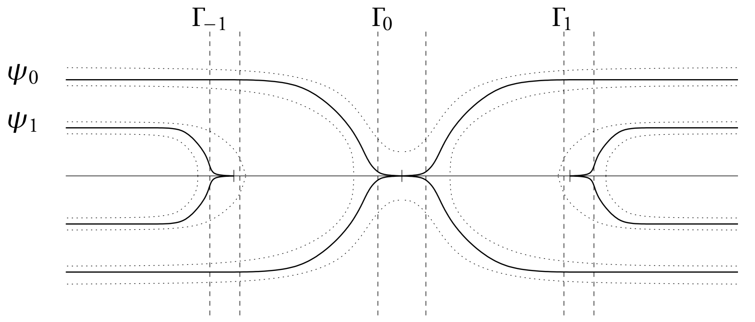

FIGURE 2.2. The cutoff functions in the case described in Remark 1.4

We may now conclude (2.8), provided that we can establish such microlocal invertibility estimates for $\Gamma_j \psi_j u$.

The invertibility estimate near $(0,0)$ has been proved in [Chr07, Chr11] for $m_1 = 1$ and in [CW13] for $m_1 > 1$. For convenience, this is restated below.

Lemma 2.2. *For $\varepsilon > 0$ sufficiently small, let $\varphi \in S(T^*\mathbb{R})$ have compact support in $\{|(x,\xi)| \leqslant \varepsilon\}$. Then, there exists $C_\varepsilon > 0$ such that*

$$\|(P-z)\varphi^w u\| \geqslant C_\varepsilon h^{2m_1/(m_1+1)} \|\varphi^w u\|, \quad z \in [1-\varepsilon, 1+\varepsilon], \tag{2.12}$$

if $m_1 > 1$. If $m_1 = 1$, then $h^{2m_1/(m_1+1)}$ is replaced by $h/(\log(1/h))$.

We need only prove the corresponding estimate near $(1,0)$. This is also used to prove Theorem 1.6.

Lemma 2.3. *For $\varepsilon > 0$ sufficiently small, let $\varphi \in S(T^*\mathbb{R})$ have compact support in $\{|(x-1,\xi)| \leqslant \varepsilon\}$. Then, there exists $C_\varepsilon > 0$ such that*

$$\|(P-z)\varphi^w u\| \geqslant C_\varepsilon h^{(4m_2+2)/(2m_2+3)} \|\varphi^w u\|, \quad z \in [C_1^{-1} - \varepsilon, C_1^{-1} + \varepsilon]. \tag{2.13}$$

The proof of this estimate proceeds through several steps. First, we rescale the principal symbol of P to introduce a calculus of two parameters. We then quantize in the second parameter which eventually will be fixed as a constant in the problem. This technique has been used in [SZ02, SZ07], [Chr07, Chr11], [CW13].

2.2. The two-parameter calculus. Before we proceed to prove Lemma 2.3, we first review some facts about the two-parameter calculus. These ideas were introduced in [SZ07], and we employ the generalizations proved in [CW13].

We set

$$S^{k,m,\tilde m}_{\alpha,\beta}(T^*(\mathbb{R}^n)) := \Big\{ a \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n)^* \times (0,1]^2) : \\ |\partial_x^\rho \partial_\xi^\gamma a(x,\xi;h,\tilde h)| \leqslant C_{\rho\gamma} h^{-m} \tilde h^{-\tilde m} \left(\frac{\tilde h}{h}\right)^{\alpha|\rho|+\beta|\gamma|} \langle \xi \rangle^{k-|\gamma|} \Big\},$$

when $\alpha \in [0,1]$ and $\beta \leq 1-\alpha$. Throughout, we take $\tilde{h} \geq h$. We abbreviate $S^{0,0,0}_{\alpha,\beta}$ by $S_{\alpha,\beta}$. The focus shall be on the marginal case $\alpha+\beta = 1$. In particular, even in this marginal case, we have that $a \in S^{k,m,\tilde{m}}_{\alpha,\beta}$ and $b \in S^{k',m',\tilde{m}'}_{\alpha,\beta}$ implies that

$$\mathrm{Op}_h^w(a) \circ \mathrm{Op}_h^w(b) = \mathrm{Op}_h^w(c)$$

for some symbol $c \in S^{k+k',m+m',\tilde{m}+\tilde{m}'}_{\alpha,\beta}$.

We also have the following expansion. This is from [SZ07, Lemma 3.6] in the case that $\alpha = \beta = \frac{1}{2}$, and from [CW13] in the more general case.

Lemma 2.4. *Suppose that $a, b \in S_{\alpha,\beta}$, and that $c^w = a^w \circ b^w$. Then,*

(2.14)
$$c(x,\xi) = \sum_{k=0}^{N} \frac{1}{k!}\left(\frac{ih}{2}\sigma(\mathrm{D}_x,\mathrm{D}_\xi;\mathrm{D}_y,\mathrm{D}_\eta)\right)^k a(x,\xi)b(y,\eta)\Big|_{\substack{x=y\\ \xi=\eta}} + e_N(x,\xi),$$

where, for some M,

(2.15)
$$|\partial^\gamma e_N| \leqslant C_N h^{N+1} \times \sum_{\gamma_1+\gamma_2=\gamma} \sup_{\substack{(x,\xi)\in T^*\mathbb{R}^n\\ (y,\eta)\in T^*\mathbb{R}^n}} \sup_{\substack{|\rho|\leqslant M\\ \rho\in\mathbb{N}^{4n}}} |\Gamma_{\alpha,\beta,\rho,\gamma}(\mathrm{D})(\sigma(\mathrm{D}))^{N+1}a(x,\xi)b(y,\eta)|,$$

where $\sigma(\mathrm{D}) = \sigma(\mathrm{D}_x,\mathrm{D}_\xi;\mathrm{D}_y,\mathrm{D}_\eta)$ as usual, and

$$\Gamma_{\alpha,\beta,\rho,\gamma}(\mathrm{D}) = (h^\alpha \partial_{(x,y)}, h^\beta \partial_{(\xi,\eta)}))^\rho \,\partial^{\gamma_1}_{(x,\xi)}\,\partial^{\gamma_2}_{(y,\eta)}.$$

With the scaling of coordinates

(2.16)
$$(x,\xi) = \mathcal{B}(X,\Xi) = ((h/\tilde{h})^\alpha X, (h/\tilde{h})^\beta \Xi),$$

it follows that if $a \in S^{k,m,\tilde{m}}_{\alpha,\beta}$, then $a \circ \mathcal{B} \in S^{k,m,\tilde{m}}_{0,0}$. Moreover, the unitary operator $T_{h,\tilde{h}}u(X) = (h/\tilde{h})^{n\alpha/2}u((h/\tilde{h})^\alpha X)$ relates the quantizations

(2.17)
$$\mathrm{Op}_{\tilde{h}}^w(a\circ\mathcal{B})T_{h,\tilde{h}}u = T_{h,\tilde{h}}\,\mathrm{Op}_h^w(a)u.$$

2.3. Proof of Lemma 2.3. Because of the cutoff φ^w, we are working microlocally in $\{|(x-1,\xi)| \leqslant \varepsilon\}$. We notice that it suffices to demonstrate (2.13) for $P-z$ replaced by $Q_1 = P - h^2V_1 - z$, as V_1 is bounded in this region, and

$$\frac{4m_2+2}{2m_2+3} < 2.$$

Let $q_1 = \xi^2 + A^{-2} - z$ be the principal symbol of Q_1. Applying Taylor's theorem about $x = 1$ to A^{-2}, we have

$$q_1 = \xi^2 - \frac{c_2}{C_1^2}(x-1)^{2m_2+1}(1 + \tilde{a}(x)) - z_1,$$

where $z_1 = z - C_1^{-1} \in [-\varepsilon, \varepsilon]$ and $\tilde{a}(x) = \mathcal{O}(|x-1|^1)$. For our specific example, this error is $\mathcal{O}(|x-1|^{2m_2+1})$, but more generally, when one merely assumes the order of vanishing at the critical point, the error is as we have listed. The Hamilton vector field H associated with the symbol q_1 is

$$\mathsf{H} = 2\xi\,\partial_x + \left((2m_2+1)\frac{c_2}{C_1^2}(x-1)^{2m_2} + \mathcal{O}(|x-1|^{2m_2+1})\right)\partial_\xi.$$

We introduce the new variables

$$X - 1 = \frac{x-1}{(h/\tilde{h})^{\alpha}}, \quad \Xi = \frac{\xi}{(h/\tilde{h})^{\beta}},$$

where

$$\alpha = \frac{2}{2m_2+3}, \quad \beta = \frac{2m_2+1}{2m_2+3},$$

and, as above, we use $\mathcal{B}$ to denote the map $\mathcal{B}(X-1, \Xi) = (x-1, \xi)$. In these new coordinates, we record that

$$\begin{aligned} \mathsf{H} &= (h/\tilde{h})^{(2m_2-1)/(2m_2+3)} \\ &\quad\times\left(2\Xi\,\partial_X + (2m_2+1)\frac{c_2}{C_1^2}(X-1)^{2m_2}\,\partial_\Xi + \mathcal{O}((h/\tilde{h})^{\alpha}|X-1|^{2m_2+1})\,\partial_\Xi\right). \end{aligned} \tag{2.18}$$

We recall the definition (2.9) of $\Lambda(r)$, and we similarly set

$$\Lambda_2(r) = 1 + \int_{-\infty}^{r} \langle t\rangle^{-1-\varepsilon_0}\,\mathrm{d}t.$$

Then, for a cutoff function $\chi(s)$ which is equal to one for $|s| < \delta_1$ and vanishes for $|s| > 2\delta_1$, we introduce

$$a(x,\xi;h,\tilde{h}) = \Lambda(\Xi)\Lambda_2(X-1)\chi(x-1)\chi(\xi),$$

where $\delta_1 > 0$ is another parameter which will be fixed shortly. As $\tilde{h} \geq h$, we have that $|\partial_X^{\ell_1}\partial_\Xi^{\ell_2} a| \leqslant C_{\ell_1,\ell_2}$ for any $\ell_1, \ell_2 \geq 0$. We compute

$$\mathsf{H}(a) = (h/\tilde{h})^{(2m_2-1)/(2m_2+3)} g(x,\xi;h,\tilde{h}) + r(x,\xi;h,\tilde{h}),$$

where

$$g = \chi(x-1)\chi(\xi)\Big(2\Lambda(\Xi)\Xi\langle X-1\rangle^{-1-\varepsilon_0} + (2m_2+1)\frac{c_2}{C_1^2}(X-1)^{2m_2}\langle\Xi\rangle^{-1-\varepsilon_0}\Lambda_2(X-1)(1+\mathcal{O}(|x-1|^1))\Big) \tag{2.19}$$

and

$$\operatorname{supp} r \subset \{|x-1| > \delta_1\} \cup \{|\xi| > \delta_1\}.$$

We first seek to show that the following lemma from [CW13] may be applied to g.

Lemma 2.5. *Let a real-valued symbol* $\tilde{g}(x,\xi;h)$ *satisfy*

$$\tilde{g}(x,\xi;h) = \begin{cases} c(\xi^2+x^{2m})(1+r_2), & \xi^2+x^2 \leqslant 1, \\ b(x,\xi;h), & \xi^2+x^2 \geqslant 1, \end{cases}$$

where $c > 0$ *is constant,* $r_2 = \mathcal{O}_{S_{\alpha,\beta}}(\delta_1)$*, and* $b > 0$ *is elliptic. Then, there exists* $c_0 > 0$ *such that*

$$\langle \mathrm{Op}_h^w(\tilde{g})u, u\rangle \geqslant c_0 h^{2m/(m+1)}\|u\|_{L^2}^2$$

for h *sufficiently small.*

In the sequel, we shall only be applying the above to functions which are microlocally cutoff to the set where $\chi(x-1)\chi(\xi) \equiv 1$. As the errors off this set will be $\mathcal{O}(h^\infty)$, we shall assume that $|x-1| \leq \delta_1$ and $|\xi| \leq \delta_1$ throughout this discussion.

On the set where $|(X-1,\Xi)| \leqslant 1$, we have $\Lambda(\Xi) \sim \Xi$, $\Lambda_2(X-1) \sim 1$ and $\langle X-1\rangle^{-1-\varepsilon_0} \sim 1$. Thus, the term g of $\mathsf{H}(a)$, a term given in (2.19), is bounded below by a multiple of $\Xi^2 + (X-1)^{2m_2}$.

We next shall consider $|(X-1,\Xi)| \geqslant 1$. Since $\operatorname{sgn}\Lambda(s) = \operatorname{sgn}(s)$, when $|\Xi| \geqslant \max(|X-1|^{1+\varepsilon_0}, \frac{1}{4})$, then

$$g \geqslant 2\Lambda(\Xi)\langle X-1\rangle^{-1-\varepsilon_0}\Xi \gtrsim \frac{|\Xi|}{\langle\Xi\rangle} \geqslant C > 0.$$

For $|X-1|^{1+\varepsilon_0} \geqslant \max(|\Xi|, \frac{1}{4})$, we have

$$g \geqslant C'\langle\Xi\rangle^{-1-\varepsilon_0}\Lambda_2(X-1)(X-1)^{2m_2} \gtrsim |X-1|^{-(1+\varepsilon_0)^2}|X-1|^{2m_2} \geqslant C'' > 0,$$

provided that $(1+\varepsilon_0)^2 < 2m_2$. In the region of interest $|(X-1,\Xi)| \geqslant 1$, the larger of $|\Xi|$ and $|X-1|^{1+\varepsilon_0}$ is assuredly greater than $\frac{1}{4}$ if $\varepsilon_0 > 0$ is sufficiently small. Hence, we have shown that $g \geqslant C > 0$ in $\{\Xi^2 + (X-1)^2 \geqslant 1\}$.

Recapping, we have found that

$$\mathsf{H}(a) = (h/\tilde{h})^{(2m_2-1)/(2m_2+3)} g + r$$

with

$$r = \mathcal{O}_{S_{\alpha,\beta}}((h/\tilde{h})^{(2m_2-1)/(2m_2+3)}((h/\tilde{h})^{\alpha}|\Xi| + (h/\tilde{h})^{\beta}|X-1|^{2m_2}))$$

supported as above, and

$$g(X,\Xi;h) = \begin{cases} c(\Xi^2 + (X-1)^{2m_2})(1+r_2), & \Xi^2 + (X-1)^2 \leqslant 1, \\ b, & \Xi^2 + (X-1)^2 \geqslant 1, \end{cases}$$

where $c > 0$ is a constant, $r_2 = \mathcal{O}_{S_{\alpha,\beta}}(\delta_1)$, and $b > 0$ is elliptic.

By translating, using the blowdown map $\mathcal{B}$, and relating the quantizations as in the previous section, we may use Lemma 2.5 to obtain a similar bound on g.

Lemma 2.6. *For g given by* (2.19) *and $\tilde{h} > 0$ sufficiently small, there exists $c > 0$ such that*

$$\| \operatorname{Op}_h^w(g \circ \mathcal{B}^{-1}) \|_{L^2 \to L^2} > c\tilde{h}^{2m_2/(m_2+1)}$$

uniformly as $h \downarrow 0$.

The proof of this lemma follows exactly as that in [CW13] and is, thus, omitted.

Before completing the proof of Lemma 2.3, we need the following lemma about the lower-order terms in the expansion of the commutator of Q_1 and a^w.

Lemma 2.7. *The symbol expansion of $[Q_1, a^w]$ in the h-Weyl calculus is of the form*

$$\begin{aligned} [Q_1, a^w] = \operatorname{Op}_h^w \bigg(\Big(\frac{ih}{2} \sigma(\mathrm{D}_x, \mathrm{D}_\xi; \mathrm{D}_y, \mathrm{D}_\eta) \Big) \\ \times (q_1(x,\xi)a(y,\eta) - q_1(y,\eta)a(x,\xi)) \big|_{x=y,\xi=\eta} + e(x,\xi) + r_3(x,\xi) \bigg), \end{aligned}$$

where r_3 is supported in $\{|(x,\xi) \geqslant \delta_1\}$, and e satisfies

$$\begin{aligned} \| \operatorname{Op}_h^w(e) \|_{L^2 \to L^2} \leqslant C\tilde{h}^{(2m_2+7)/(2m_2+3) - 2m_2/(m_2+1)} h^{(4m_2+2)/(2m_2+3)} \\ \times (\| \operatorname{Op}_h^w(g \circ \mathcal{B}^{-1}) \|_{L^2 \to L^2} + \mathcal{O}(\tilde{h}^{2+2m_2/(m_2+1)})), \end{aligned}$$

with g given by (2.19).

Proof. Since everything is in the Weyl calculus, only the odd terms in the exponential composition expansion are nonzero. In accordance with Lemma 2.4, we set

$$e(x,\xi) = \chi(x-1)\chi(\xi)\sum_{k=1}^{m_2-1}\frac{2}{(2k+1)!}\left(\frac{ih}{2}\sigma(\mathrm{D})\right)^{2k+1} \\ \times q_1(x,\xi)\Lambda((\tilde{h}/h)^{\beta}\eta)\Lambda_2((\tilde{h}/h)^{\alpha}(y-1))\big|_{x=y,\xi=\eta} \\ + \chi(\xi)\chi(x-1)e_{2m_2}(x,\xi).$$

Here, we have extracted the terms in the expansion where derivatives fall on the cutoff $\chi(\eta)$ of a, as these terms have supports compatible with r_3. For convenience, however, e_{2m_2} denotes the full error in the expansion of $[Q_1, a^w]$.

Recalling that $q_1(x,\xi) = \xi^2 - (x-1)^{2m_2+1}(1+\tilde{a}(x))$, it follows that

$$\begin{aligned}\tilde{e}_k &:= h^{2k+1}\chi(x-1)\chi(\xi)\sigma(\mathrm{D})^{2k+1}q_1(x,\xi)\Lambda((\tilde{h}/h)^{\beta}\eta)\Lambda_2((\tilde{h}/h)^{\alpha}(y-1))\Big|_{\substack{x=y\\ \xi=\eta}} \\ &= h^{2k+1}\chi(x-1)\chi(\xi)\mathrm{D}_x^{2k+1}q_1(x,\xi)\mathrm{D}_\eta^{2k+1}\Lambda((\tilde{h}/h)^{\beta}\eta)\Lambda_2((\tilde{h}/h)^{\alpha}(y-1))\Big|_{\substack{x=y\\ \xi=\eta}} \\ &= ch^{2k+1}(x-1)^{2m_2+1-(2k+1)}(1+\mathcal{O}((x-1)^{2m_2+1}))(\tilde{h}/h)^{(2k+1)\beta}\Lambda^{(2k+1)} \\ &\qquad \times ((\tilde{h}/h)^{\beta}\xi)\Lambda_2((\tilde{h}/h)^{\alpha}(x-1))\chi(x-1)\chi(\xi)\end{aligned}$$

for $1 \le k \le m_2 - 1$.

In order to estimate e, we first estimate each $\tilde{e}_k$, $1 \le k \le m_2 - 1$, using conjugation to the two-parameter calculus. We have

$$\begin{aligned}\|\operatorname{Op}_h^w(\tilde{e}_k)u\|_{L^2} &= \|T_{h,\tilde{h}}\operatorname{Op}_h^w(\tilde{e}_k)T_{h,\tilde{h}}^{-1}T_{h,\tilde{h}}u\|_{L^2} \\ &\leqslant \|T_{h,\tilde{h}}\operatorname{Op}_h^w(\tilde{e}_k)T_{h,\tilde{h}}\|_{L^2\to L^2}\|u\|_{L^2}\end{aligned}$$

since $T_{h,\tilde{h}}$ is unitary. We recall $T_{h,\tilde{h}}\operatorname{Op}_h^w(\tilde{e}_k)T_{h,\tilde{h}}^{-1} = \operatorname{Op}_{\tilde{h}}^w(\tilde{e}_k\circ\mathcal{B})$, and note that

$$\tilde{e}_k\circ\mathcal{B} = ch^{2k+1}(h/\tilde{h})^{(2m_2+1-(2k+1))\alpha-(2k+1)\beta}(X-1)^{2m_2+1-(2k+1)} \\ \times (1+\mathcal{O}((x-1)^1))\Lambda^{(2k+1)}(\Xi)\Lambda_2(X-1)\chi(x-1)\chi(\xi),$$

which can be estimated by

$$Ch^{(4m_2+2)/(2m_2+3)}\tilde{h}^{(2m_2+7)/(2m_2+3)}\tilde{h}^{2(k-1)} \\ \times (X-1)^{(2m_2+1)-(2k+1)}\Lambda^{(2k+1)}(\Xi)\chi(x-1)\chi(\xi).$$

On the set $\{|X-1| \leqslant 1\}$, we have that

$$(X-1)^{(2m_2+1)-(2k+1)}\Lambda^{(2k+1)}(\Xi)\chi(x-1)\chi(\xi)$$

is bounded, and thus, by Lemma 2.6,

$$\|\operatorname{Op}_{\tilde{h}}^w(\tilde{e}_k)\|_{L^2\to L^2} \le C\tilde{h}^{-2m_2/(m_2+1)}\|\operatorname{Op}_h^w(g\circ\mathcal{B}^{-1})\|_{L^2\to L^2}.$$

While on $\{|X-1|\geqslant 1\}$, we have $\tilde{e}_k \leq g$, and thus

$$\begin{aligned}\|\operatorname{Op}_h^w(\tilde{e}_k)\|_{L^2\to L^2} &\leq \|\operatorname{Op}_h^w(g)\|_{L^2\to L^2} + O(\tilde{h}^2)\\ &\leq \|\operatorname{Op}_h^w(g\circ\mathcal{B}^{-1})\|_{L^2\to L^2} + O(\tilde{h}^2).\end{aligned}$$

For e_{2m_2}, by the standard L^2-continuity theorem of h-pseudodifferential operators, it suffices to estimate a finite number of derivatives of the error e_{2m_2}. We note the bound of Lemma 2.4:

$$|\partial^\gamma e_{2m_2}| \leqslant Ch^{2m_2+1}\sum_{\gamma_1+\gamma_2=\gamma}\sup_{\substack{(x,\xi)\in T^*\mathbb{R}^n\\(y,\eta)\in T^*\mathbb{R}^n\\ \rho\in\mathbb{N}^{4n},\,|\rho|\leq M}}|\Gamma_{\alpha,\beta,\rho,\gamma}(\mathrm{D})(\sigma(\mathrm{D}))^{2m_2+1}q_1(x,\xi)a(y,\eta)|.$$

We have

$$\begin{aligned}&(\sigma(\mathrm{D}))^{2m_2+1}q_1(x,\xi)a(y,\eta)\\ &\quad= c(1+\mathcal{O}(x-1)^1)\chi(y-1)\Lambda((\tilde{h}/h)^\alpha(y-1))\mathrm{D}_\eta^{2m_2+1}[\Lambda((\tilde{h}/h)^\beta\eta)\chi(\eta)].\end{aligned}$$

The last factor is $\mathcal{O}((\tilde{h}/h)^{(2m_2+1)\beta})$. Moreover, the derivatives $h^\beta\partial_\eta$ and $h^\alpha\partial_y$ preserve the order of h and increase the order of $\tilde{h}$, while the other derivatives lead to higher powers of $h/\tilde{h}\ll 1$. Thus, it follows that $|\partial^\gamma(\chi(x-1)\chi(\xi)e_{2m_2})|$ is

$$\mathcal{O}(h^{(4m_2+2)/(2m_2+3)}\tilde{h}^{(2m_2+1)^2/(2m_2+3)}),$$

and therefore, when combined with Lemma 2.4, it satisfies the given bound. □

We now complete the proof of Lemma 2.3. We set $v=\varphi^w u$ where φ has support in the set $\{\chi(x)\chi(\xi)=1\}$, and particularly, away from the support of r_3.

Then, Lemmas 2.6 and 2.7 yield

$$\begin{aligned}i\langle[Q_1,a^w]v,v\rangle &= h\langle\operatorname{Op}_h^w(\mathsf{H}(a))v,v\rangle + \langle\operatorname{Op}_h^w(e)u,u\rangle + \mathcal{O}(h^\infty)\|v\|_{L^2}^2\\ &= h(h/\tilde{h})^{(2m_2-1)/(2m_2+3)}\langle\operatorname{Op}_h^w(g\circ\mathcal{B}^{-1})v,v\rangle + \langle\operatorname{Op}_h^w(e)u,u\rangle + \mathcal{O}(h^\infty)\|v\|_{L^2}^2\\ &= h^{(4m_2+2)/(2m_2+3)}(\tilde{h}^{-(2m_2-1)/(2m_2+3)} + \mathcal{O}(\tilde{h}^{(2m_2+7)/(2m_2+3)-2m_2/(m_2+1)}))\\ &\qquad\times\langle\operatorname{Op}_h^w(g\circ\mathcal{B}^{-1})v,v\rangle + (\mathcal{O}(h^\infty)+\mathcal{O}(\tilde{h}^{2+2m_2/(m_2+1)}))\|v\|_{L^2}^2\\ &\geqslant Ch^{(4m_2+2)/(2m_2+3)}\tilde{h}^{1+4/(2m_2+3)-2/(m_2+1)}\|v\|_{L^2}^2,\end{aligned}$$

for $\tilde{h}$ sufficiently small. The Schwarz inequality and the L^2-continuity theorem for h-pseudodifferential operators guarantee that

$$|\langle[Q_1,a^w]v,v\rangle| \leqslant C\|Q_1v\|_{L^2}\|v\|_{L^2},$$

and thus also guarantee the desired bound with $1\gg\tilde{h}>0$ fixed.

3. Quasimodes

We end by constructing quasimodes near $(1,0)$ in phase space, and use these to saturate the estimate of Proposition 2.1, and hence that of Theorem 1.1. The proofs follow from straightforward modifications of those in [CW13], and so we provide only a brief description.

Quasimodes were already constructed near $(0,0)$ in [CW13]. We focus only on the construction near the inflection point. We let

$$\tilde{P} = -h^2\partial_x^2 - c_2(x-1)^{2m_2+1}$$

near $x = 1$, and construct quasimodes that are localized to a small neighborhood of $x = 1$.

We set

$$\gamma = \frac{4m+2}{2m+3},$$

let $E = (\alpha + i\beta)h^\gamma$ where $\alpha, \beta > 0$ and are independent of h, and set

$$\varpi(x) = \int_1^x (E + c_2(y-1)^{2m_2+1})^{1/2}\,\mathrm{d}y,$$

where the branch of the square root is chosen to have positive imaginary part. Letting

$$u(x) = (\varpi')^{-1/2}e^{i\varpi/h},$$

we see that

$$(h\mathrm{D})^2u = (\varpi')^2u + fu,$$

where

$$f = -h^2\left(\frac{3}{4}(\varpi')^{-2}(\varpi'')^2 - \frac{1}{2}(\varpi')^{-1}\varpi'''\right).$$

Straightforward modifications of the proof contained in [CW13, Lemma 3.1] yield the following result.

Lemma 3.1. *The phase function ϖ satisfies the following properties:*

(i) *There exists $C > 0$ independent of h such that*

$$|\operatorname{Im}\varpi| \leqslant Ch.$$

In particular, if $|x-1| \leqslant Ch^{\gamma/(2m_2+1)}$, then $|\operatorname{Im}\varpi| \leqslant C'$ for some $C' > 0$ independent of h.

(ii) *There exists $C > 0$ independent of h such that, if $\delta > 0$ is sufficiently small and $|x-1| \leqslant \delta h^{\gamma/(2m_2+1)}$, then*

$$C^{-1}h^{\gamma/2} \leqslant |\varpi'(x)| \leqslant Ch^{\gamma/2}.$$

(iii) *We have*

$$\begin{aligned}
\varpi' &= (E + c_2(x-1)^{2m_2+1})^{1/2},\\
\varpi'' &= \frac{1}{2}c_2(2m_2+1)(x-1)^{2m_2}(\varpi')^{-1},\\
\varpi''' &= \Big(\frac{1}{2}c_2(2m_2+1)(2m_2)(E(x-1)^{2m_2-1} + c_2(x-1)^{4m_2})\\
&\qquad - \frac{1}{4}c_2^2(2m_2+1)^2(x-1)^{4m_2}\Big)(\varpi')^{-3}.
\end{aligned}$$

In particular, there are constants $C_{m_2,1}, C_{m_2,2}$ such that

$$f = -h^2\left(C_{m_2,1}(x-1)^{4m_2} + C_{m_2,2}E(x-1)^{2m_2-1}\right)(\varpi')^{-4}.$$

From this, we obtain that $|u(x)| \sim |\varphi'|^{-1/2}$ for all x. We localize u by setting

$$\mu = \delta h^{\gamma/(2m_2+1)}, \quad 0 < \delta \ll 1,$$

by fixing $\chi(s) \in C_c^\infty(\mathbb{R})$ so that $\chi \equiv 1$ for $|s| \leqslant 1$ and $\operatorname{supp}\chi \subset [-2,2]$, and by letting

$$\tilde{u}(x) = \chi((x-1)/\mu)u(x).$$

More calculations, which are again in the spirit of those contained in [CW13], show that $\|\tilde{u}\|_{L^2}^2 \sim h^{(1-2m_2)/(2m_2+3)}$ and $(h\mathrm{D})^2\tilde{u} = (\varpi')^2\tilde{u} + R$, where

$$R = f\tilde{u} + [(h\mathrm{D})^2, \chi((x-1)/\mu)]u.$$

Moreover, the remainder satisfies $\|R\|_{L^2} = \mathcal{O}(h^\gamma)\|\tilde{u}\|_{L^2}$.

This quasimode can then be used to saturate the local smoothing estimates near the inflection point. Again, we refer the interested reader to the proof in [CW13, Theorem 3], which provides the following result.

Theorem 3.2. *Let $\varphi_0(x,\theta) = e^{ik\theta}\tilde{u}(x)$, where $\tilde{u} \in C_c^\infty(\mathbb{R})$ was constructed above. We let $h = |k|^{-1}$, where $|k|$ is taken sufficiently large. Suppose ψ solves*

$$\begin{cases} (\mathrm{D}_t - \tilde{\Delta})\psi = 0,\\ \psi|_{t=0} = \varphi_0. \end{cases}$$

Then, for any $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi \equiv 1$ on $\operatorname{supp}\tilde{u}$ and $A > 0$ sufficiently large, independent of k, there exists a constant $C_0 > 0$ independent of k, such that

$$\int_0^{|k|^{-4/(2m_2+3)}/A} \|\langle \mathrm{D}_\theta\rangle \chi\psi\|_{L^2}^2\,\mathrm{d}t \geqslant C_0^{-1}\|\langle \mathrm{D}_\theta\rangle^{(2m_2+1)/(2m_2+3)}\varphi_0\|_{L^2}^2. \tag{3.1}$$

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