The Strauss conjecture on black hole backgrounds

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The Strauss conjecture

$$\begin{cases} \Box u = |u|^p, \quad \Box = \partial_t^2 - \Delta, \\ u(0,\,\cdot\,) = \varepsilon f, \quad \partial_t u(0,\,\cdot\,) = \varepsilon g \\ f, g \in C_c^\infty \end{cases}$$

- For which p is there a global solution for ε sufficiently small?
  - On $\mathbb{R}_+ \times \mathbb{R}^n$, there are global solutions for ε sufficiently small provided $p > p_c$ where p_c is the positive root of

$$(n-1)p_c^2 - (n+1)p_c - 2 = 0.$$

  - When $n = 3$, $p_c = 1 + \sqrt{2}$.

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Theorem

On Kerr backgrounds with $a \ll M$ solutions to the above wave equation exist globally provided the Cauchy data (f, g) are compactly supported, ε is sufficiently small, and $p > 1 + \sqrt{2}$.

Strategy of proof

- Need to find spaces X, Y so that
  - $\|u\|_X \lesssim \varepsilon + \|\Box u\|_Y$
  - $\||u|^p\|_Y \lesssim \|u\|_X^p$
- Near infinity, the metric is a small perturbation of the Minkowski metric, and weighted Strichartz estimates, such as those derived in the Minkowski setting, may be applied.
- In the compact region about the event horizon (and region containing trapping), we use localized energy estimates.

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Localized energy estimates

For u solving

$$\begin{cases} \Box u = F, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \quad n \geq 3 \\ u(0,\,\cdot\,) = f, \quad \partial_t u(0,\,\cdot\,) = g, \end{cases}$$

we have

$$\|u'\|_{L^\infty_t L^2_x} + \sup_{j\geq 0}\Big(2^{-j/2}\|u'\|_{L^2_{t,x}(\mathbb{R}_+\times\langle x\rangle\approx 2^j)} + 2^{-3j/2}\|u\|_{L^2_{t,x}(\mathbb{R}_+\times\langle x\rangle\approx 2^j)}\Big)$$
$$\lesssim \|u'(0,\,\cdot\,)\|_2 + \inf_{F=F_1+F_2}\Big(\|F_1\|_{L^1_t L^2_x} + \sum_{j\geq 0} 2^{j/2}\|F_2\|_{L^2_{t,x}(\mathbb{R}_+\times\langle x\rangle\approx 2^j)}\Big).$$

Weighted Strichartz estimate

On $\mathbb{R}_+ \times \mathbb{R}^3$, suppose u solves $\Box u = F$ with vanishing Cauchy data, $2 \leq p \leq \infty$, and s satisfies

$$\frac{1}{2} - \frac{1}{p} < s < \frac{1}{2}.$$

Then we have

$$\left\| |x|^{\frac{3}{2} - \frac{4}{p} - s} u \right\|_{L_t^p L_r^p L_\omega^2} \lesssim \left\| |x|^{-\frac{1}{2} - s} F \right\|_{L_t^1 L_r^1 L_\omega^2}.$$

Spaces

Set $s = 3/2 - 2/(p-1)$, $\alpha = \frac{2(p-2)}{p(p-1)}$, and

$$\|u\|_X = \sum_{|\gamma|\le 2} \Big(\|r^{-\alpha}\chi Z^\gamma u\|_{L^p_{t,r}L^2_\omega} + \|\langle r\rangle^{-3/2-\delta}\partial^\gamma u\|_{L^2_{t,r,\omega}} + \|\partial^\gamma \partial u\|_{L^\infty_t L^2_{r,\omega}} \Big)$$

$$\|F\|_Y = \sum_{|\gamma|\le 2} \Big(\|r^{-\alpha p}\chi^p Z^\gamma F\|_{L^1_{t,r}L^2_\omega} + \|Z^\gamma F\|_{L^1_t L^2_{r,\omega}} \Big)$$

Closing the loop: $\||u|^p\|_Y \lesssim \|u\|_X^p$

- $\|r^{-\alpha p}\chi^p|u|^p\|_{L^1_{t,r}} \lesssim \|r^{-\alpha}\chi u\|^p_{L^p_{t,r}}$
  - Sobolev embedding on $\mathbb{S}^2$ completes the proof in this case.
- To show, e.g.,

$$\|(\chi u)^{p-1}\chi Z^\gamma u\|_{L^1_t L^2_{r,\omega}} \lesssim \Big(\sum_{|\beta|\le 2} \|r^{-\alpha}\chi Z^\beta u\|_{L^p_{t,r}L^2_\omega}\Big)^p,$$

we apply weighted Sobolev estimates such as

$$\|v\|_{L^\infty(|x|\ge R+1)} \lesssim \sum_{|\gamma|\le 2} \|r^{-2/p} Z^\gamma v\|_{L^p_r L^2_\omega(r\ge R)}.$$

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Weighted Strichartz estimates

- A duality argument gives

$$\|u\|_{L_t^\infty L_x^2} + \sup_j 2^{-j/2}\|u\|_{L^2_{t,x}(\mathbb{R}_+\times\langle x\rangle\approx 2^j)} \lesssim \|\Box u\|_{L_t^1\dot{H}_x^{-1}}.$$

- Interpolation with the original yields, for $0 < s < 1$,

$$\|u\|_{L_t^\infty \dot{H}_x^s} + \sup_j 2^{(-1/2-s)j}\|u\|_{L^2_{t,x}(\mathbb{R}_+\times\langle x\rangle\sim 2^j)} \lesssim \|\Box u\|_{L_t^1\dot{H}^{s-1}}.$$

- Combining with a trace theorem $\|r^{3/2-s}f\|_{L_r^\infty L_\omega^2} \lesssim \|f\|_{\dot{H}^s}$, $1/2 < s < 3/2$ gives

$$\|r^{3/2-s}w\|_{L^\infty_{t,r}L^2_\omega} \lesssim \|\Box u\|_{L_t^1\dot{H}^{s-1}}$$

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Weighted Strichartz estimates

- Interpolating again

$$\|r^{3/2-4/p-s}u\|_{L^p_{t,r}L^2_\omega} \lesssim \|F\|_{L^1_t \dot{H}^{s-1}_x}, \quad 1/2-1/p < s < 1.$$

- Finally, applying the dual to the trace theorem,

$$\|r^{3/2-4/p-s}u\|_{L^p_{t,r}L^2_\omega} \lesssim \|r^{-1/2-s}F\|_{L^1_{t,r}L^2_\omega}, \quad 1/2-1/p < s < 1/2.$$

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