Low regularity local well-posedness for quasilinear Schrödinger equations

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(joint work with Jeremy Marzuola, Daniel Tataru)

This is a report on the results of [5, 6] that concern quasilinear Schrödinger equations

$$(1)\qquad \begin{aligned} i\partial_t u + g^{jk}(u, \nabla u)\partial_j\partial_k u &= F(u, \nabla u), \quad u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}^M, \\ u(0) &= u_0. \end{aligned}$$

Here, in the case of quadratic interactions, we assume

$$g^{jk}(y, z) = I_{d\times d} + O(|y| + |z|), \quad F(y, z) = O(|y|^2 + |z|^2), \quad \text{near } (0, 0),$$

while for cubic interactions we instead have

$$g^{jk}(y, z) = I_{d\times d} + O(|y|^2 + |z|^2), \quad F(y, z) = O(|y|^3 + |z|^3), \quad \text{near } (0, 0).$$

We shall also permit the ultrahyperbolic case where the identity matrix above can be replaced by one with a different signature.

Local well-posedness, which means existence, uniqueness, and continuous dependence on the initial data, was proved in [3]. See, also, [4] for the ultrahyperbolic case. The current studies seek to improve upon the necessary regularity, in the small data regime, that must be assumed on the initial datum in order to have local well-posedness.

A key difficulty arises in connection with the Mizohata integrability condition (see, e.g., [2]), which requires that for the linear problem

$$(i\partial_t + \Delta_g)v = A_i(x)\partial_i v$$

to be L^2 well-posed one must have that the real part of $A(x)$ is integrable along the Hamiltonian flow of Δ_g. In order to guarantee such for the linearization of (1) in the quadratic case, it is not enough to work in Sobolev spaces. Additional decay must be assumed. The approach of [3], [4] was to work in weighted spaces and to use an "artificial viscosity method".

Our approach is inspired by [1], and decay is imposed by assuming a summability over cubes. To this end, we let $\mathcal{Q}_j$ be a partition of $\mathbb{R}^d$ into cubes of sidelength 2^j. For an associated smooth partition $\{\chi_Q\}$ and for a Banach space U, we define

$$\|u\|^p_{l^p_j U} = \sum_{Q\in\mathcal{Q}_j} \|\chi_Q u\|^p_U.$$

In particular, for the quadratic case, we measure the initial datum in the norm

$$\|u\|^2_{l^1 H^s} = \sum_{j\geq 0} 2^{2js}\|S_j u\|^2_{l^1_j L^2},$$

where $\sum_{j\geq 0} S_j(D) = 1$ is a Littlewood-Paley partition. On the other hand, in the cubic case, upon linearization, the Mizohata condition is trivially satisfied due to energy estimates, and here it suffices to take data in standard Sobolev spaces.

Indeed, the main results are:

Theorem 1. *There exists $\varepsilon > 0$ and a Banach space S so that if the initial datum u_0 satisfies $\|u_0\|_S < \varepsilon$, then (1) is locally well-posed in S on the time interval $I = [0,1]$.*

- *(Quadratic [5]) For the quadratic case, we take $S = l^1 H^s$ for $s > \frac{d}{2} + 3$.*
- *(Cubic [6]) For the cubic case, we take $S = H^s$ for $s > \frac{d+5}{2}$.*

The iteration spaces and main linear estimate are inspired by the classical local smoothing / energy estimate

$$\sup_{R\geq 0} R^{-1/2} \|D^{1/2} e^{it\Delta} f\|_{L^2_{t,x}([0,1]\times\{\langle x\rangle \approx R\})} + \|e^{it\Delta} f\|_{L^\infty_t L^2_x} \lesssim \|f\|_{L^2}.$$

Indeed, we set

$$\|u\|_{X_j} = 2^{j/2}\Big(\sup_l \sup_{Q\in\mathcal{Q}_l} 2^{-l/2} \|u\|_{L^2_{t,x}([0,1]\times Q))}\Big) + \|u\|_{L^\infty_t L^2_x}$$

and

$$\|u\|^2_{l^1 X^s} = \sum_{j\geq 0} 2^{2js} \|S_j u\|^2_{l^1_j X_j}.$$

The dual-type spaces in which we measure the nonlinearities satisfy the following bound

$$\|f\|_{Y_j} \lesssim \inf_{f = 2^{j/2} f_1 + f_2} \Big[\Big(\inf_l \sum_{Q\in\mathcal{Q}_l} 2^{l/2} \|f_1\|_{L^2_{t,x}([0,1]\times Q)}\Big) + \|f_2\|_{L^1_t L^2_x}\Big]$$

and

$$\|f\|^2_{l^1 Y^s} = \sum_{j\geq 0} 2^{2js} \|S_j f\|^2_{l^1_j Y_j}.$$

The principal linear estimate, in the quadratic case, then states that solutions to

$$\begin{cases} (i\partial_t + \partial_k g^{kl} \partial_l) u + V\cdot\nabla u = h, \\ u(0) = u_0 \end{cases}$$

satisfy

$$\|u\|_{l^1 X^\sigma} \lesssim \|u_0\|_{l^1 H^\sigma} + \|h\|_{l^1 Y^\sigma}$$

provided that

$$\|g - I\|_{l^1 X^s} \ll 1, \quad \|V\|_{l^1 X^{s-1}} \ll 1, \quad s > \frac{d}{2} + 2, \quad 0 \leq \sigma \leq s.$$

The proof relies on a positive commutator argument. One first passes to a frequency localized set up. A wedge decomposition is employed to further restrict attention to the case that the solution is localized in frequency in a small cone about each coordinate axis. This permits us to trivially handle the ultrahyperbolic case as one need only modify the sign of the multiplier corresponding to those directions where $g(0)$ is negative. About the x_1 axis, e.g., the multiplier is chosen as

$$i2^j \mathcal{M} = m(2^{-l} x_1)\partial_1 + \partial_1 m(2^{-l} x_1),$$

where $m'(s) = \psi^2(s)$, $\psi \in \mathcal{S}$ and ψ is localized to frequencies $\lesssim 1$ and $\psi \sim 1$ on $|s| \leq 1$. Repeating similarly in the other coordinate directions, we obtain via a positive commutator argument an estimate in a cube of sidelength 2^l centered at the origin. Taking a supremum over translates and scales, up to introducing the l^1 summability, the estimate is obtained. The latter is incorporated by introducing cutoffs adapted to a scale that is slightly larger than desired (say M times the desired scale), commuting with the operator, and using M^{-1} as a small parameter to bootstrap the commutators. As M is independent of the frequency scale, passing to the desired summability is trivial.

The iteration to show local well-posedness is then closed with multilinear and Moser-type estimates. For example, we have

$$\|uv\|_{l^1 Y^\sigma} \lesssim \|u\|_{l^1 X^{s-1}} \|v\|_{l^1 X^{\sigma-1}}$$

provided $s > \frac{d}{2} + 2$ and $0 \leq \sigma \leq s$. These are proved using the definitions of the spaces and a usual analysis of the low-high, high-low, and high-high trichotomy.

The arguments in the cubic cases proceed quite similarly. The primary difference is that the l^1 summability is replaced by l^2 summability. In particular, on the initial data, we have $l^2 H^s \approx H^s$.

References

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