Long time existence for nonlinear wave equations in exterior domains

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These results focus on long-time existence for nonlinear wave equations with small initial data in exterior domains. In particular, the focus is on nonlinear equations where dependence on the solution u, rather than just its first and second derivatives, is permitted at the lowest order.

Two primary problems are studied. In both cases, we shall fix a bounded obstacle $\mathcal{K} \subset \mathbb{R}^n$ with smooth boundary and examine the wave equation exterior to it.

The first problem is an analog of the Strauss conjecture and regards equations of the form

$$\begin{cases} \Box u := (\partial_t^2 - \Delta)u = |u|^p, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \backslash \mathcal{K}, \\ (Bu)|_{\partial\mathcal{K}} = 0, \\ u(0,\,\cdot\,) = f, \quad \partial_t u(0,\,\cdot\,) = g. \end{cases} \tag{1}$$

The obstacle is assumed to be nontrapping, and B is either the identity operator or the normal derivative. The Cauchy data are assumed to satisfy the relevant compatibility condition. The question concerns the values of p for which there is global existence if the data are taken sufficiently small.

In the absence of a boundary, the question of existence was resolved in [4], [13], where it is shown that $p > p_c$, where p_c is the larger root of $(n-1)p_c^2 - (n+1)p_c - 2 = 0$, guarantees the existence of global solutions. A more thorough history of the problem can also be found therein. The first progress on this problem in exterior domains has come in [2] ($n = 4$), which is a joint work with Du, Sogge, and Zhou, and [5] ($n = 3, 4$), which is a collaboration with Hidano, Smith, Sogge, and Zhou. The key tool is a class of weighted Strichartz estimates

$$\Big\| |x|^{\frac{n}{2} - \frac{n+1}{p} - \gamma} u \Big\|_{L^p_{t,r} L^2_\omega} \lesssim \|u'(0,\,\cdot\,)\|_{\dot{H}^{\gamma-1}} + \Big\| |x|^{-\frac{n}{2}+1-\gamma} \Box u \Big\|_{L^1_{t,r} L^2_\omega}, \tag{2}$$

$$2 \le p \le \infty, \quad \frac{1}{2} - \frac{1}{p} < \gamma < \frac{1}{2}, \tag{3}$$

which are obtained by interpolating between a trace lemma and a localized energy estimate which is proved using Plancherel's theorem. The index $\gamma = \frac{n}{2} - \frac{2}{p-1}$ corresponds precisely to $p_c < p < \frac{n+3}{n-1}$. Moreover, for this γ, we have

$$\Big\| |x|^{-\frac{n}{2}+1-\gamma} |u|^p \Big\|_{L^1} = \Big\| |x|^{\frac{n}{2} - \frac{n+1}{p} - \gamma} u \Big\|_{L^p}^p.$$

Thus, provided the nonlinearity is sufficiently regular to allow for the Sobolev embeddings which are needed in the angular variables, which corresponds to $n \le 4$, an iteration can be closed to show small data global existence. In order to obtain analogs of (2) when there is a boundary, arguments akin to those of [12], [1], and [9] may be adapted to permit the weighted spaces.

The second class of nonlinear problems which are examined concern quasilinear wave equations with nonlinearities vanishing to second order

$$
\begin{cases} \Box u = Q(u, u', u''), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \backslash \mathcal{K}, \\ u|_{\partial \mathcal{K}} = 0, \\ u(0,\,\cdot\,) = f, \quad \partial_t u(0,\,\cdot\,) = g. \end{cases} \tag{4}
$$

The novelty here is nonlinearity's dependence on the solution u at the quadratic level, rather than just on u' and u''. For data of size ε, the boundaryless studies [8], [6] establish the goal of showing a c/ε^2 $(n = 3)$ and $\exp(c/\varepsilon)$ $(n = 4)$ lower bound on the lifespan respectively.

For star-shaped obstacles $\mathcal{K}$ and data satisfying the compatibility conditions, the three dimensional lifespan was proved in [3] and in four dimensions, in the author's joint work [2] with Du, Sogge, and Zhou. The key estimate of [2] is a localized energy estimate that corresponds to the $p = 2$, $\gamma = 0$ endpoint of (2), for which there is a logarithmic blow-up in the length of the time-interval T.

This estimate is combined with a localized energy estimate for perturbations of the d'Alembertian from [10], which holds exterior to star-shaped obstacles. One then iterates in the fashion developed in [7] which uses localized energy estimates to permit one to show long-time existence based on decay in $|x|$ rather than decay in t.

If the additional hypothesis that $(\partial_u^2 Q)(0,0,0) = 0$ is imposed on the nonlinearity, which disallows a u^2 term, then a longer lifespan is expected. On $\mathbb{R}^4$, [6] showed that such equations have global solutions for sufficiently small data, and exterior to a star-shaped obstacle, an analogous result was proved by the author and Sogge [11]. Here, one utilizes the $p = 2$ version of (2) with $\gamma = 2\delta$. This is combined with an estimate for equations with forcing terms in divergence form, which says that if $\Box u = \sum_{j=0}^4 a_j \partial_j G$ with vanishing initial data, then

$$
\|\langle x\rangle^{-1/2-\delta} u\|_{L^2_{t,x}} \lesssim \|G(0,\,\cdot\,)\|_{\dot{H}^{\gamma-1}} + \int_0^T \|G(t,\,\cdot\,)\|_2 \, dt \tag{5}
$$

provided $0 < \delta < 1/2$. The proof uses techniques which were also employed in [6], [8], but rather than applying them to the energy inequality, they are instead applied to a localized energy estimate

$$
\|\langle x\rangle^{-1/2-\delta} u'\|_{L^2_{t,x}} \lesssim \|u'(0,\,\cdot\,)\|_2 + \int_0^T \|\Box u(t,\,\cdot\,)\|_2 \, dt. \tag{6}
$$

The method of iteration can be illustrated by studying the boundaryless semilinear equation

$$
\Box u = u\partial_t u + (\partial_t u)^2 = \frac{1}{2}\partial_t(u^2) + (\partial_t u)^2
$$

for smooth initial data of size ε. We set

$$
M(T) = \sum_{|\alpha| \le 10} \Big(\|\langle x\rangle^{-1/2-\delta} (Z^\alpha u)'\|_{L^2_{t,x}} + \|\langle x\rangle^{-1/2-2\delta} Z^\alpha u\|_{L^2_{t,x}} \Big),
$$

where Z denotes the set of vector fields $\{\partial_k, x_i\partial_j - x_j\partial_i\}$ where $0 \le k \le 4$ and $1 \le i < j \le 4$. Utilizing that $[Z, \Box] = 0$, we may apply (2), (5), and (6) to see that

$$M(T) \lesssim \varepsilon + \sum_{|\alpha|+|\beta|\le 10} \| |x|^{-1-2\delta} (Z^\alpha u)'(Z^\beta u)' \|_{L^1_{t,r} L^2_\omega} + \sum_{\substack{|\alpha|+|\beta|\le 10 \\ |\mu|,|\nu|\le 1}} \int_0^T \|\partial^\mu Z^\alpha u \partial^\nu Z^\beta u\|_2 \, dt.$$

The key is to notice that only quadratic terms involving u' rather than just u appear in the second term in the right. An application of Sobolev embedding on the sphere and the Schwarz inequality to the second term in the right and an application of a standard weighted Sobolev inequality (see [7] and the references therein) which provides $O(\langle x\rangle^{-(n-1)/2})$ decay to the third term in the right shows that $M(T) \lesssim \varepsilon + (M(T))^2$, from which it is easy to construct an iteration to show global existence.

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